PHD Filters of Higher Order in Target Number

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The multitarget recursive Bayes nonlinear filter is the theoretically optimal approach to multisensor-multitarget detection, tracking, and identification. For applications in which this filter is appropriate, it is likely to be tractable for only a small number of targets. In earlier papers we derived closed-form equations for an approximation of this filter based on propagation of a first-order multitarget moment called the probability hypothesis density (PHD). In a recent paper, Erdinc, Willett, and Bar-Shalom argued for the need for a PHD-type filter which remains first-order in the states of individual targets, but which is higher-order in target number. In this paper we show that this is indeed possible. We derive a closed-form cardinalized PHD (CPHD) filter, which propagates not only the PHD but also the entire probability distribution on target number.

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I. INTRODUCTION

The multitarget recursive Bayes filter is the theoretically optimal approach to multisensor-multitarget detection, tracking, and identification. Given a time-sequence $Z^{(k)}: Z_1,\ldots,Z_k$ of multitarget measurement-sets, it propagates the Bayes multitarget posterior via an alternating sequence of predictor (time-update) and corrector (data-update) steps (see Section IIF):

$$\cdots \longrightarrow f_{k|k}(X \mid Z^{(k)}) \underset{\text{predictor}}{\longrightarrow} f_{k+1|k}(X \mid Z^{(k)})$$

$$\underset{\text{corrector}}{\longrightarrow} f_{k+1|k+1}(X \mid Z^{(k+1)}) \longrightarrow \cdots.$$

For applications in which this filter is appropriate—i.e., those, such as low SNR in which conventional approaches such as multi-hypothesis correlation (MHC) perform poorly—it is likely to be tractable only for a small number of targets. Consequently, it may prove to be of limited practical interest in the absence of drastic but principled approximation strategies.

In single-target problems the computationally fastest approximate filtering approach is the constant-gain Kalman filter, of which the alpha-beta filter is the most familiar instance. Such filters propagate a first-order statistical moment (the posterior expectation) in the place of the posterior distribution. In an earlier paper [22] we proposed an analogous strategy for multitarget systems: propagation of a first-order multitarget moment. This moment, the probability hypothesis density (PHD) $D_{k|k}(\mathbf{x} \mid Z^{(k)})$, is uniquely defined by the property that its integral in any region of state space is the expected number of targets in that region. We derived recursive Bayes filter equations for the PHD that account for nonconstant probability of detection, Poisson false alarm processes, and appearance, spawning, and disappearance of targets (see Section IIG):

$$\cdots \longrightarrow D_{k|k}(\mathbf{x} \mid Z^{(k)}) \underset{\text{predictor}}{\longrightarrow} D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$$

$$\underset{\text{corrector}}{\longrightarrow} D_{k+1|k+1}(\mathbf{x} \mid Z^{(k+1)}) \longrightarrow \cdots.$$

We also showed that the PHD is a best-fit approximation of the multitarget posterior in an information-theoretic sense.

The PHD filter has inspired much recent research, to be summarized in Section IIG1. Some of this research [20, 31] suggests, somewhat surprisingly, that the PHD filter may have practical utility beyond the cluster tracking and group-target tracking applications for which it was originally intended. Nevertheless, its potential limitations have been evident to those who, like the author, are aware of the superior performance of second-order (Kalman) over first-order (alpha-beta) filters in the single-target case. At an early stage we investigated the possibility of second-order generalizations of the PHD filter which would propagate a covariance density $C_{k|k}(\mathbf{x},\mathbf{y} \mid Z^{(k)})$ as well as the PHD [23, pp. 155–156]. We concluded that such filters were unlikely to be computationally tractable under realistic sensing conditions.

In a recent conference paper [16, sect. 5.2] Erdinc, Willett, and Bar-Shalom pointed out that the PHD filter's greatest potential strength is also arguably its most obvious potential limitation. On the one hand, it propagates an estimate $N_{k|k}$ of the expected number $\bar{n}$ of targets in the scene, thus providing a potentially powerful decluttering technique. On the other hand, the value of $N_{k|k}$ is very unstable in the presence of missed detections and/or significantly large false alarm densities. These authors argued that the instability of $N_{k|k}$ is a direct consequence of the fact that the PHD is only first-order. Using a simple single-target example, they showed that the instability of $N_{k|k}$ is attributable to the PHD filter's linearization $\bar{n}_{k+1|k+1} = (1-p_D)\cdot \bar{n}_{k+1|k}$ of the following nonlinear formula for expected number of targets [16, eq. (37)] at the data-update (corrector) step:

$$\bar{n}_{k+1|k+1} = \frac{(1-p_D)\cdot \bar{n}_{k+1|k}}{1-\bar{n}_{k+1|k}\cdot p_D}. \tag{1}$$

Here $p_D < 1$ is (constant) probability of detection and, under these assumptions, $\bar{n}_{k+1|k}$ is the predicted track probability (probability of existence) of the target. If $\bar{n}_{k+1|k} = 1$, for example, (1) estimates $\bar{n}_{k+1|k+1} = 1$ whereas the PHD filter loses information by estimating $\bar{n}_{k+1|k+1} = 1 - p_D$.

At the conference we noted that, in lieu of a tractable second-order solution, a heuristic "fix" (maintaining a windowed running average of $N_{k|k}$) results in a reasonably stable and accurate estimate of target number. However, this remedy will be effective only if the rate of target appearance and/or disappearance is not too great compared with the data-update rate.

Erdinc et al. further proposed that the PHD filter should be generalized to provide not only $N_{k|k}$ but also an estimate $\sigma^2_{k|k}$ of the variance of $N_{k|k}$. What was not evident at the time was that this suggestion constituted a different way of looking at second-order multitarget approximation. They were proposing a search for a filter which remains first-order in the states of individual targets, but which is higher-order in target number. Such a partial higher-order solution could sidestep much of the computational intractability of a full second-order approximation. The primary purpose of this paper is to demonstrate that, contrary to our initial expectations, partial higher order multitarget approximation is indeed possible.

This realization led us to another insight. The instability of $N_{k|k}$ is attributable not just to loss of second-order information, but also to a well-known fact about state estimation. The PHD filter employs the expected a posteriori (EAP) state estimator, i.e., the expected value $\bar{n}_{k|k} = \sum_{n\geq 0} n\cdot p_{k|k}(n \mid Z^{(k)})$ of the probability distribution $p_{k|k}(n \mid Z^{(k)})$ of the number of targets (hereafter called the cardinality distribution). EAP estimation typically produces unstable and inaccurate state-estimates under lower SNR conditions. The reason is that the minor modes of the posterior distribution, which are induced by false alarms and thus tend to be highly random, erratically perturb the expected value away from the target-induced primary mode. The maximum a posteriori (MAP) estimator, by way of contrast, ignores minor modes and locks onto the more stable and accurate major mode. For this reason it is more commonly employed in the application of nonlinear filters.

Consequently, more stable and accurate estimates of target number could be possible if the entire distribution $p_{k|k}(n \mid Z^{(k)})$ were recursively propagated along with the PHD and if MAP estimates were derived from it. A second purpose of this paper is to show that this also is possible.

We derive closed-form predictor and corrector equations for what we call the cardinalized PHD (CPHD) filter, (61)–(65). This is a generalization of the PHD filter which propagates not only $D_{k|k}(\mathbf{x} \mid Z^{(k)})$ but also $p_{k|k}(n \mid Z^{(k)})$ and its probability generating function $G_{k|k}(x \mid Z^{(k)})$ via a time-update step

$$\begin{array}{ccccc} \cdots\longrightarrow & \begin{matrix} p_{k|k} \\ G_{k|k}\end{matrix} & \longrightarrow & \begin{matrix} p_{k+1|k} \\ G_{k+1|k}\end{matrix} & \\ \cdots\longrightarrow & \begin{matrix} D_{k|k} \\ N_{k|k}\end{matrix} & \underset{\text{predictor}}{\longrightarrow} & \begin{matrix} D_{k+1|k} \\ N_{k+1|k}\end{matrix} & \end{array} \tag{2}$$

and a data-update step

$$\begin{array}{cccc} \begin{matrix} p_{k+1|k} \\ G_{k+1|k}\end{matrix} & \longrightarrow & \begin{matrix} p_{k+1|k+1} \\ G_{k+1|k+1}\end{matrix} & \longrightarrow\cdots \\ & & \uparrow\downarrow & \\ \begin{matrix} D_{k+1|k} \\ N_{k+1|k}\end{matrix} & \underset{\text{corrector}}{\longrightarrow} & \begin{matrix} D_{k+1|k+1} \\ N_{k+1|k+1}\end{matrix} & \longrightarrow\cdots. \end{array} \tag{3}$$

The pair of vertical arrows at the corrector step indicates that the filter for p and G and the filter for D and N are mutually coupled. On the one hand, the formula for $D_{k+1|k+1}(\mathbf{x})$ requires $G_{k+1|k}(x)$ and its derivatives of all orders $G^{(i)}_{k+1|k}(x)$, $i \geq 0$. On the other hand, the formulas for $G_{k+1|k+1}(x)$ and $p_{k+1k+1}(n)$ require $D_{k+1|k}(\mathbf{x})$ and its integral $N_{k+1|k} = \int D_{k+1|k}(\mathbf{x})d\mathbf{x}$. The CPHD filter recursion therefore requires, for every $k \geq 0$, explicit formulas for $N_{k+1|k}$ and $N_{k+1|k+1}$, for $D_{k+1|k}(\mathbf{x})$ and $D_{k+1|k+1}(\mathbf{x})$, for $p_{k+1|k}(n)$ and $p_{k+1|k+1}(n)$, and for all derivatives of $G_{k+1|k}(x)$ and $G_{k+1|k+1}(x)$.

The CPHD filter further generalizes the PHD filter in that the sensor false alarm process can be a general "independent and identically distributed (IID) cluster process" [11, pp. 122–123, 145] rather than strictly Poisson. That is, the physical distribution of false alarms is still governed by a single intensity density, but the probability distribution of their number can now be arbitrary. In other respects the CPHD filter is less general than the PHD filter in that spawning of targets by other targets can no longer be explicitly modeled.

The CPHD filter is likely to be computationally tractable in many situations of practical interest (see Section IIID). Vo, Vo, and Cantoni have implemented and successfully tested it and compared its performance with that of the PHD filter and the joint probabilistic data association (JPDA) filter (see Section V). There are limits to its computability, however, since its corrector step has computational order $O(n \cdot m^2 \cdot \log^2 m)$. A search for partial second-order approximations of order $O(n \cdot m)$ is therefore advisable.

The paper is organized as follows. Section II reviews those basic concepts of multitarget statistics required to understand what follows. The CPHD filter is introduced in Section III. Mathematical proofs have been relegated to Section IV. A summary and conclusions may be found in Section V.

II. BASIC MULTITARGET STATISTICS

Finite-set statistics (FISST) [18, 21, 26–29] is, in part, an "engineering-friendly" formulation of point process theory [11] for information fusion applications. This section reviews those aspects of FISST necessary to understand the remainder of the paper.

We introduce set integrals and multitarget probability density functions in Section IIA, and probability hypothesis densities or PHDs (first-order multitarget moments) in Section IIB. Probability generating functionals (PGFLs) and their functional derivatives are introduced in Sections IIC and IID, respectively. Poisson processes and IID cluster processes, families of multitarget processes central to this paper, are described in Section IIE. The general multitarget recursive Bayes filter is introduced in Section IIF and its first-order multitarget moment approximation, the PHD filter, in Section IIG. Since derivation of the predictor and corrector equations for the CPHD filter requires the same techniques used to derive those for the PHD filter, we review these in Section IIH.

A. Multitarget Probability Density Functions

Assume that states of individual targets have the form $\mathbf{x} = (\mathbf{x}', c)$ where $\mathbf{x}'$ denotes the continuous kinematic state variables (e.g., position, velocity, etc.) and c the discrete state variables (e.g., target type, target identity, track label, etc.). Since the number n of targets in a scenario can be any nonnegative integer, and since targets have no intrinsic ordering, the state X of a multitarget system is most naturally represented as a finite set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the target state-vectors. In a Bayesian formulation the unknown state variable X must be a random variable, i.e., a random finite set Ξ in which not only the $\mathbf{x}_1, \ldots, \mathbf{x}_n$ randomly vary, but so does n.

The multitarget probability distribution of Ξ is a probability distribution $f_\Xi(X)$ on the finite-set variable X—which is to say, it must sum to unity over all possible multitarget states X. This requires a set integral

$$\int f_\Xi(X)\delta X \triangleq f(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_\Xi(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}) d\mathbf{x}_1 \cdots d\mathbf{x}_n \tag{4}$$

which accounts for the random variability of n as well as of the $\mathbf{x}_1, \ldots, \mathbf{x}_n$. A multitarget probability distribution must satisfy $\int f_\Xi(X)\delta X = 1$. Multitarget distributions differ from single-target distributions in that the units of measurement of $f_\Xi(X)$ vary with the number $|X|$ of elements in X.

The cardinality distribution is the probability distribution $p_\Xi(n) \triangleq \Pr(|\Xi| = n)$ of the number of elements in the random finite subset Ξ. It is given by

$$p_\Xi(n) = \int_{|X|=n} f_\Xi(X)\delta X \tag{5}$$

$$\triangleq \frac{1}{n!} \int f_\Xi(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}) d\mathbf{x}_1 \cdots d\mathbf{x}_n. \tag{6}$$

In general, measurements collected from multiple targets can be arbitrary in number as well as in value, and do not have any natural ordering. Thus, multitarget measurements in general will have the form of a finite set $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$ of ordinary measurements and the random multitarget measurement will be a random finite subset Σ of measurements.

Rather than having to develop parallel but otherwise identical formulations of multi-object statistics for states and for measurements, it is more efficient to develop a single formulation of probability distributions $f_\Psi(Y)$ of random finite subsets Ψ with instantiations $\Psi = Y$ drawn from arbitrary (e.g., state or measurement) spaces.

B. First-Order Multitarget Moments

A naïve definition of the expected value of a random finite set Ψ would be $\bar{\Psi} = \int Y \cdot f_\Psi(Y)\delta Y$.

However, this integral is mathematically nonsensical since addition $Y + Y'$ of finite subsets Y, Y' cannot be usefully defined. Consequently, one must instead select a transformation $Y \mapsto \tau(Y)$ which converts finite subsets Y into vectors in some vector space. This transformation should also transform unions into sums: $\tau(Y \cup Y') = \tau(Y) + \tau(Y')$ whenever $Y \cap Y' = \emptyset$. Then one can define an "indirect" expected value as $\mathrm{E}[\tau(\Psi)]$. We select $\tau(Y) = \delta_Y$ where

$$\delta_Y(\mathbf{y}) \triangleq \sum_{\mathbf{w} \in Y} \delta_{\mathbf{w}}(\mathbf{y}) \tag{7}$$

and where $\delta_{\mathbf{w}}(\mathbf{y})$ denotes the Dirac delta function concentrated at $\mathbf{w}$. Then

$$D_\Psi(\mathbf{y}) \triangleq \mathrm{E}[\delta_\Psi(\mathbf{y})] = \int \delta_Y(\mathbf{y}) \cdot f_\Psi(Y)\delta Y \tag{8}$$

is a multitarget analog of the concept of expected value. Note that it is a conventional density function. It is not a probability density since the integral of $D_\Psi(\mathbf{y})$ in any region S is the expected number of objects of Ψ in that region:

$$\int_S D_\Psi(\mathbf{y}) d\mathbf{y} = \mathrm{E}[|S \cap \Psi|]. \tag{9}$$

The density $D_\Psi(\mathbf{y})$ is called the "intensity density" or PHD of Ψ (or of $f_\Psi(Y)$). Intuitively speaking, $D_\Psi(\mathbf{y})$ represents the zero-probability event $\Pr(\mathbf{y} \in \Psi)$ in the same way that a conventional probability density $f_{\mathbf{Y}}(\mathbf{y})$ represents the zero-probability event $\Pr(\mathbf{Y} = \mathbf{y})$. If the elements of $Y = X$ are states, the value $D_\Xi(\mathbf{x})$ can be interpreted as the track density at $\mathbf{x}$.

C. Probability Generating Functionals

Signal processing engineers are familiar with the power of integral transform methods: Fourier transforms, Laplace transforms, z-transforms, etc. [41]. Mathematical operations which are complicated in the signal domain, such as integrals, often become simple when converted to the frequency domain using a suitable integral transform. In ordinary statistics it is similarly useful to employ transform concepts such as characteristic function, probability generating function, etc. We have likewise found it useful to introduce transform methods into multitarget analysis [22]. In Section IIC1 we summarize the properties of the probability generating function (PGF) of a random integer. In Section IIC2 we generalize this concept to the PGFL of a random finite set.

1) *Probability Generating Functions*: Let $p_J(n) = \Pr(J = n)$ be the probability distribution of a random nonnegative integer J. The PGF of J is, whenever defined on $0 \le y \le 1$,

$$G_J(y) \triangleq \mathrm{E}[y^J] = \sum_{n=0}^{\infty} y^n \cdot p_J(n). \tag{10}$$

The PGF gets its name from the fact that

$$p_J(n) = \frac{1}{n!} G_J^{(n)}(0) \tag{11}$$

where $G_J^{(n)}(y)$ denotes the nth derivative of $G_J(y)$. Note that $0 \le G_J(y) \le 1$ and $G_J(1) = 1$. (If $p_J(0), p_J(1), \ldots, p_J(n), \ldots$ were the components of a discrete-time signal then $G_J(z^{-1})$ would be its z-transform [41, p. 61, eq. (4–6)].)

If n_J is the expected value and σ_J^2 the variance of J, it is easy to show that

$$n_J = G_J^{(1)}(1), \qquad \sigma_J^2 = G_J^{(2)}(1) - n_j^2 + n_J. \tag{12}$$

If $G_1(y)$ and $G_2(y)$ are two PGFs with respective expected values n_1, n_2 and variances σ_1^2, σ_2^2, then $G_{12}(y) = G_1(y) \cdot G_2(y)$ is a PGF and its expected value and variance are, respectively,

$$n_{12} = n_1 + n_2, \qquad \sigma_{12}^2 = \sigma_1^2 + \sigma_2^2. \tag{13}$$

The derivatives of all orders of $G_{12}(y)$ can be computed using the generalized product rule

$$G_{12}^{(n)}(y) = \sum_{i=0}^{n} C_{n,i} \cdot G_1^{(i)}(y) \cdot G_2^{(n-i)}(y) \tag{14}$$

where $C_{n,i} = n!/i!(n-i)!$ is the binomial coefficient.

2) *Probability Generating Functionals*: Let $f_\Psi(Y)$ be the probability distribution of a finite random set Ψ, i.e., $\int f_\Psi(Y)\delta Y = 1$. Its PGFL is[1]

$$G_\Psi[h] \triangleq \mathrm{E}[h^\Psi] = \int h^Y \cdot f_\Psi(Y)\delta Y \tag{15}$$

where for any "test function" $h(\mathbf{y})$ with $0 \le h(\mathbf{y}) \le 1$, h^Y is defined as

$$h^Y \triangleq \begin{cases} 1 & \text{if} \quad Y = \emptyset \\ h(\mathbf{y}_1) \cdots h(\mathbf{y}_n) & \text{if} \quad Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\} \end{cases} \tag{16}$$

and where $n = |Y|$. Note that $0 \le G_\Psi[h] \le 1$ and $G_\Psi[1] = 1$. Roughly speaking, $f_\Psi \mapsto G_\Psi[h]$ is a kind of generalized z-transform, in which functions $f_\Psi(Y)$ on the finite-set domain are transformed into functions on the test-function domain.

Let $p_\Psi(n) = \int_{|Y|=n} f_\Psi(Y)\delta Y$ be the cardinality distribution of $f(Y)$, as defined in (5), and let $G_\Psi(y)$ be its PGF. If $h(\mathbf{y}) = y$ is a constant function with value y, then

$$G_\Psi[y] = G_\Psi(y). \tag{17}$$

[1] This is a simplified definition. Our definition of functional derivatives below in (18) will be nonsensical unless PGFLs are defined on test functions $h(\mathbf{x})$ which can involve Dirac delta functions. We ignore this complexity here. See [22, p. 1161, sect. A] for a more careful definition.

D. Functional Derivatives of PGFLs

The first functional derivative of a PGFL $G_\Psi[h]$ with respect to $\mathbf{y}$ is

$$\frac{\delta G_\Psi}{\delta \mathbf{y}}[h] \triangleq \lim_{\varepsilon \searrow 0} \frac{G_\Psi[h+\varepsilon\delta_{\mathbf{y}}] - G_\Psi[h]}{\varepsilon} \tag{18}$$

where $\delta_{\mathbf{y}}(\mathbf{y}')$ denotes the Dirac delta function concentrated at $\mathbf{y}$ and where, for each fixed h, the transformation $\mathbf{y} \mapsto (\delta G_\Psi/\delta \mathbf{y})[h]$ is assumed to be linear and continuous.

Iterated functional derivatives are defined recursively. If $Y = \{\mathbf{y}_1,\ldots,\mathbf{y}_n\}$ with $|Y| = n$,

$$\frac{\delta G_\Psi}{\delta Y}[h] \triangleq \frac{\delta^n G_\Psi}{\delta \mathbf{y}_1 \cdots \delta \mathbf{y}_n}[h] \tag{19}$$

$$\triangleq \frac{\delta}{\delta \mathbf{y}_n}\left(\frac{\delta^{n-1} G_\Psi}{\delta \mathbf{y}_1 \cdots \delta \mathbf{y}_{n-1}}[h]\right). \tag{20}$$

For completeness, the functional derivative with respect to the empty set $Y = \emptyset$ is defined as

$$\frac{\delta G_\Psi}{\delta \emptyset}[h] \triangleq G_\Psi[h]. \tag{21}$$

The importance of functional derivatives arises from the fact that they allow multi-object density functions to be computed from PGFLs. Let $f_\Psi(Y)$ be a multi-object density function and let $G_\Psi[h]$ be its PGFL. Then it can be shown that the following multi-object analog of (11) is true:

$$f_\Psi(Y) = \frac{\delta G_\Psi}{\delta Y}[0]. \tag{22}$$

The PHD $D_\Psi(\mathbf{y})$, or first multi-object moment of $f_\Psi(Y)$, can be computed from the PGFL of $f_\Psi(Y)$:

$$D_\Psi(\mathbf{y}) = \frac{\delta G_\Psi}{\delta \mathbf{y}}[1]. \tag{23}$$

Functional derivatives obey "turn the crank" rules analogous to those of elementary calculus, e.g.:

Linear Functional Rule: Let $s(\mathbf{y})$ be a density function and $s[h] \triangleq \int h(\mathbf{w}) \cdot s(\mathbf{w}) d\mathbf{w}$ its corresponding linear functional. Then $\delta^n s/(\delta \mathbf{y}_1 \cdots \delta \mathbf{y}_n)[h] = 0$ if $n > 1$ and

$$\frac{\delta s}{\delta \mathbf{y}}[h] = s(\mathbf{y}). \tag{24}$$

Product Rule:

$$\frac{\delta(G_1 \cdot G_2)}{\delta \mathbf{y}}[h] = \frac{\delta G_1}{\delta \mathbf{y}}[h] \cdot G_2[h] + G_1[h] \cdot \frac{\delta G_2}{\delta \mathbf{y}}[h]. \tag{25}$$

General Product Rule: Equation (25) can be generalized to iterated functional derivatives as follows [18, p. 151]:

$$\frac{\delta(G_1 \cdot G_2)}{\delta Y}[h] = \sum_{W \subseteq Y} \frac{\delta G_1}{\delta (Y-W)}[h] \cdot \frac{\delta G_2}{\delta W}[h] \tag{26}$$

where the summation is taken over all subsets W of Y (including $W = \emptyset$ and $W = Y$).

Chain Rule: For any ordinary real-valued function $f(y_1,\ldots,y_n)$ of real variables $y_1,\ldots,y_n$,

$$\frac{\delta}{\delta \mathbf{y}} f(G_1[h],\ldots,G_n[h]) = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(G_1[h],\ldots,G_n[h]) \cdot \frac{\delta G_i}{\delta \mathbf{y}}[h]. \tag{27}$$

Power Rule: Substituting $n = 1$ and $f(y) = y^n$ into the chain rule, we get

$$\frac{\delta}{\delta \mathbf{y}} G[h]^n = n \cdot G[h]^{n-1} \cdot \frac{\delta G}{\delta \mathbf{y}}[h]. \tag{28}$$

D. IID Cluster Processes

This section introduces multitarget processes which are central to our discussions later. The multitarget Poisson process has PGFL

$$G[h] = e^{\mu \cdot f[h] - \mu}. \tag{29}$$

The cardinality distribution of a Poisson process is a Poisson distribution $p(n) = e^{-\mu}\mu^n/n!$ with PGF $G(y) = e^{\mu y - \mu}$. A Poisson process on state space describes a multitarget system in which the physical distribution of targets is described by the single probability density $f(\mathbf{x})$ and target number by a Poisson distribution.

Let $G(y) = \sum_{n=0}^{\infty} y^n \cdot p(n)$ be the PGF of an arbitrary probability distribution $p(n)$ on object number. An IID cluster process [11, pp. 122–123, 145] is one which has a PGFL of the form

$$G[h] = G(f[h]). \tag{30}$$

An IID cluster process on state space describes a multitarget system in which the physical distribution of targets is described by the single probability density $f(\mathbf{x})$ and target number by an arbitrary distribution $p(n)$.

The functional derivatives of an IID cluster process are easily computed using the chain and linear functional rules for functional derivatives:

$$\frac{\delta G}{\delta \mathbf{y}_1 \cdots \delta \mathbf{y}_n}[h] = G^{(n)}(f[h]) \cdot f(\mathbf{y}_1) \cdots f(\mathbf{y}_n). \tag{31}$$

This formula will allow us to derive closed-form formulas for the CPHD corrector step, (61)–(65).

F. The Multitarget Bayes Filter

Let $Z^{(k)} : Z_1,\ldots,Z_k$ be a time-sequence of multitarget measurement-sets. The general multitarget Bayes recursive filter is defined by the equations

$$f_{k+1|k}(X \mid Z^{(k)}) = \int f_{k+1|k}(X \mid X') \cdot f_{k|k}(X' \mid Z^{(k)}) \delta X' \tag{32}$$

and

$$f_{k+1|k+1}(X \mid Z^{(k+1)}) = \frac{f_{k+1}(Z_{k+1} \mid X) \cdot f_{k+1|k}(X \mid Z^{(k)})}{f_{k+1}(Z_{k+1} \mid Z^{(k)})} \quad (33)$$

where the Bayes normalization factor is

$$f_{k+1}(Z_{k+1} \mid Z^{(k)}) = \int f_{k+1}(Z_{k+1} \mid X) \cdot f_{k+1|k}(X \mid Z^{(k)}) \delta X \quad (34)$$

and where the integrals are set integrals. These equations are not the straightforward generalizations of the single-target Bayes filter that they might appear to be. They require the techniques of FISST [18, 21, 26, 27, 29].

G. The PHD Filter

As was noted in the Introduction, (32) and (33) are likely to be tractable only for a small number of targets in those applications in which they are appropriate e.g., low SNR. Consequently, we proposed the PHD filter [22] as a multitarget statistical analog of the computationally fastest approximate single-target filtering approach: the constant-gain Kalman filter. This filter propagates a first-order statistical moment in the place of the multitarget posterior distribution.

In this section we review the basic ideas of the PHD filter: the prediction step (Section IIG2), correction step (Section IIG3), and estimation step (Section IIG4). We begin in Section IIG1 by briefly summarizing recent PHD filter research.

1) *Recent PHD Filter Research*: The PHD filter has usually been implemented using sequential Monte Carlo (a.k.a. particle-system) methods, as proposed by Kjellström (nee Sidenbladh) [38] and by Zajic and Mahler [55]. Instances are Erdinc, Willett, and Bar-Shalom [16], and Vo, Singh, and Doucet [49]. Vo, Singh, Doucet, and Clark have established convergence results for the particle-PHD filter [49, p. 1234, Prop. 3], [5].

Vo and Ma [45] have, under certain simplifying assumptions, devised a closed-form Gaussian-mixture implementation which greatly improves the computational efficiency of the PHD filter. This approach is inherently capable of maintaining track labels [9, 10]. Clark and Vo [10] have proved a strong L_1 uniform convergence property for the Gaussian-mixture PHD filter, in the sense that "each step in time of the PHD filter will maintain a suitable approximation error that converges to zero as the number of Gaussians in the mixture tends to infinity."

In this section we briefly summarize current PHD filter research.

Erdinc, Willett, and Bar-Shalom [17] have proposed a purely physical interpretation of both the PHD and CPHD filters.

Since the "core" PHD filter does not maintain labels for tracks from time-step to time-step, two groups of researchers have independently proposed "peak to track association" techniques for maintaining track labels with particle-PHD filters: Panta, Vo, Doucet, and Singh [31]; and Lin, Kirubarajan, and Bar-Shalom [20]. These authors have demonstrated in 1D and 2D simulations that their track-valued PHD filters can outperform conventional MHT-type techniques (i.e., significantly fewer false and dropped tracks).

Punithakumar and Kirubarajan [37] have implemented a multiple motion model version of the PHD filter, as have Pasha, Vo, Tuan, and Ma [34, 35].

Punithakumar, Kirubarajan, and Sinha [36] have devised and implemented a distributed PHD filter that addresses the problem of communicating and fusing multitarget track information from a distributed network of sensor-carrying platforms.

Balakumar, Sinha, Kirubarajan, and Reilly [2] have applied a PHD filter to the problem of tracking an unknown and time-varying number of narrowband, far-field signal sources, using a uniform linear array of passive sensors, in a highly nonstationary sensing environment.

Ahlberg, Hörling, Kjellström, Jöred, Mårtenson, Neider, Schubert, Svenson, Svensson, Undén, and Walter of the Swedish Defence Research Agency (FOI) have employed PHD filters for group-target tracking in an ambitious situation assessment simulator system called "IFD03" [38–40].

Tobias and Lanterman have applied the PHD filter to target detection and tracking using bistatic RF observations [42–44].

Clark, Bell, de Saint-Pern, and Petillot have applied the PHD filter to both 2D and 3D active-sonar problems [4, 6–9].

Ikoma, Uchino, and Maeda [19] have applied a PHD filter to the problem of tracking the trajectories of feature-points in time-varying optical images. Wang, Wu, Kassim, and Huang [54] have employed such methods to tracking groups of humans in digital video.

Zajic, Ravichandran, et al. [56] report an algorithm in which a PHD filter is integrated with a robust classifier algorithm which identifies airborne targets from high range-resolution radar (HRRR) signatures.

El-Fallah, Zatezalo, et al. [13–15, 12] have demonstrated implementations of the PHD filter-based sensor management approach described in [24].

2) *PHD Filter Predictor*: From time-step k we have in hand the PHD $D_{kk}(\mathbf{x} \mid Z^{(k)})$. We are to derive a formula for the predicted PHD $D_{k+1|k}(\mathbf{x} \mid Z^{(k+1)})$. We make the following assumptions.

a) Motion of individual targets: $f_{k+1|k}(\mathbf{x} \mid \mathbf{x}')$ is the single-target Markov transition density.

b) Disappearance of existing targets: $p_{S,k+1|k}(\mathbf{x}')$ is the probability that a target with state $\mathbf{x}'$ at time-step k will survive in time-step $k+1$, and is hereafter abbreviated as $p_S(\mathbf{x})$.

c) Target spawning: $b_{k+1|k}(X \mid \mathbf{x}')$ is the likelihood that a group of new targets with state-set X will be spawned at time-step $k+1$ by a single target that had state $\mathbf{x}'$ at time-step k; and its PHD is denoted by $b_{k+1|k}(\mathbf{x} \mid \mathbf{x}')$.

d) Appearance of completely new targets: $b_{k+1|k}(X)$ is the likelihood that new targets with state-set X will enter the scene at time-step $k+1$, and its PHD is denoted as $b_{k+1|k}(\mathbf{x})$.

Given this it can be shown that the PHD predictor step is [22, eq. (75)]:

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int F_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot D_{k|k}(\mathbf{x}')d\mathbf{x}' \tag{35}$$

where

$$F_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \stackrel{\Delta}{=} p_S(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') + b_{k+1|k}(\mathbf{x} \mid \mathbf{x}'). \tag{36}$$

3) *PHD Filter Corrector*: The PHD filter presumes the same multitarget measurement model that is employed in multitarget trackers such as MHT and JPDA. From the predictor step we have in hand the predicted PHD $D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$. At time-step $k+1$ we collect a new observation-set $Z_{k+1} = \{\mathbf{z}_1,\ldots,\mathbf{z}_m\}$ with m elements. We require a formula for the data-updated PHD $D_{k+1|k+1}(\mathbf{x} \mid Z^{(k+1)})$. Abbreviate $D_{k+1|k}(\mathbf{x}) = D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$, and $D_{k+1|k+1}(\mathbf{x}) = D_{k+1|k+1}(\mathbf{x} \mid Z^{(k+1)})$. Further,

a) Single-target measurement generation: $L_{k+1,\mathbf{z}}(\mathbf{x}) = f_{k+1}(\mathbf{z} \mid \mathbf{x})$ is the sensor likelihood function, hereafter abbreviated as $L_{\mathbf{z}}(\mathbf{x})$.

b) Probability of detection: $p_{D,k+1}(\mathbf{x})$ is the probability that an observation will be collected at time-step $k+1$ from a target with state $\mathbf{x}$, hereafter abbreviated as $p_D(\mathbf{x})$.

c) Poisson false alarms: at time-step $k+1$ the sensor collects an average number λ_{k+1} of Poisson-distributed false alarms, the spatial distribution of which is governed by the probability density $c_{k+1}(\mathbf{z})$, and these are hereafter abbreviated as λ and $c(\mathbf{z})$.

An additional simplifying assumption is required if we are to derive closed-form formulas for the corrector step:

d) Poisson multitarget prior: the predicted multitarget distribution $f_{k+1|k}(X \mid Z^{(k)})$ is approximately Poisson with PGFL $G_{k+1|k}[h] = e^{\mu s[h]-\mu}$ where $\mu \stackrel{\Delta}{=} N_{k+1|k} = \int D_{k+1|k}(\mathbf{x} \mid Z^{(k)})d\mathbf{x}$; where $s(\mathbf{x}) \stackrel{\Delta}{=} N^{-1}_{k+1|k} \cdot D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$; where $s[h] \stackrel{\Delta}{=} \int h(\mathbf{x}) \cdot s(\mathbf{x})d\mathbf{x}$; and where $D_{k+1|k}[h] = \mu s[h]$.

Given this it can be shown that the PHD corrector step is [22, eqs. (87)–(88)]:

$$D_{k+1|k+1}(\mathbf{x}) \cong L_{Z_{k+1}}(\mathbf{x}) \cdot D_{k+1|k}(\mathbf{x}) \tag{37}$$

where for any measurement-set Z,

$$L_Z(\mathbf{x}) \stackrel{\Delta}{=} 1 - p_D(\mathbf{x}) + \sum_{\mathbf{z}\in Z} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x})}{\lambda c(\mathbf{z}) + D_{k+1|k}[p_D L_{\mathbf{z}}]}. \tag{38}$$

Multiple sensors: Suppose that at time-step $k+1$ two measurement-sets $\overset{1}{Z}_{k+1}$, $\overset{2}{Z}_{k+1}$ are collected from two different sensors. The rigorous formula for the PHD corrector step appears to be too complicated to be of practical use [22, p. 1169]. The most common heuristic approach is to apply the PHD corrector step twice in succession, once for $\overset{1}{Z}_{k+1}$ and once for $\overset{2}{Z}_{k+1}$. A more rigorous approximate "pseudosensor" approach has been described in [24, p. 271–276].

4) *PHD Filter State Estimator*: Extracting multitarget state estimates from a PHD is conceptually simple in principle. The PHD provides an estimate $N_{k|k}$ of the number of targets. After rounding this to the nearest integer n, one looks for the n largest local suprema $D_1,\ldots,D_n$ of the PHD and declares the corresponding $\mathbf{x}_1,\ldots,\mathbf{x}_n$ such that $D_i = D_{k|k}(\mathbf{x}_i \mid Z^{k)})$ to be the state-estimates of the targets.

In practice, multitarget state estimation is easy or difficult depending on the technique which has been employed to implement the PHD filter. It is very easy if one employs the Gaussian-mixture approximation of Vo and Ma [45]. Furthermore, this approach is inherently capable of maintaining track labels. Multitarget state estimation is considerably more difficult if sequential Monte Carlo approximation is adopted. In this case, one common approach is to use the expectation-maximization (EM) algorithm to find the best approximation of the PHD using a weighted sum of n Gaussian distributions. The centroids of these Gaussians are then chosen as the target state estimates. Clustering techniques have also been applied [55].

H. Derivation of the PHD Filter Equations

It is necessary to review the methodology used to derive the PHD predictor and corrector equations in [22], since this same methodology is used to derive the corresponding equations for the CPHD filter. We address the PHD predictor and corrector steps in turn. (For a more detailed summary see [27, sect. 4].)

1) *Derivation of the PHD Filter Predictor*: Rewrite the multitarget predictor equation, (32), in PGFL form:

$$G_{k+1|k}[h] = \int G_{k+1|k}[h \mid X'] \cdot f_{k|k}(X' \mid Z^{(k)})\delta X' \tag{39}$$

where $G_{k+1|k}[h]$ is the PGFL of $f_{k+1|k}(X \mid Z^{(k)})$ and where $G_{k+1|k}[h \mid X'] \triangleq \int h^X \cdot f_{k+1|k}(X \mid X')\delta X$ is the PGFL of $f_{k+1|k}(X \mid X')$. Given the multitarget motion model described at the beginning of Section IIG2, one can derive a closed-form formula for $f_{k+1|k}(X \mid X')$ and then for $G_{k+1|k}[h]$ [22, pp. 1172–1173]:

$$G_{k+1|k}[h] = e_h \cdot G[(1 - p_S + p_S p_h) \cdot b_h] \tag{40}$$

where $G[h] = G_{k|k}[h]$ is the PGFL of $f_{k|k}(X' \mid Z^{(k)})$ and where

$$p_h(\mathbf{x}') \triangleq \int h(\mathbf{x}) \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}')d\mathbf{x} \tag{41}$$

$$b_h(\mathbf{x}') \triangleq \int h^X \cdot b_{k+1|k}(X \mid \mathbf{x}')\delta X \tag{42}$$

$$e_h \triangleq \int h^X \cdot b_{k+1|k}(X)\delta X. \tag{43}$$

The predicted PHD $D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$ can be determined using (23) and the formulas for the functional derivative, (24)–(28).

The CPHD predictor will be derived in the same fashion, except that $b_h(\mathbf{x}') = 1$ identically (i.e., no target spawning) and that $G[h]$ will be assumed to be an IID cluster process.

2) *Derivation of the PHD Filter Corrector*: Begin by rewriting the numerator of multitarget Bayes' rule, (33), in PGFL form:

$$F[g,h] \triangleq \int h^X \cdot G_{k+1}[g \mid X] \cdot f_{k+1|k}(X \mid Z^{(k)})\delta X \tag{44}$$

where

$$G_{k+1}[g \mid X] \triangleq \int g^Z \cdot f_{k+1}(Z \mid X)\delta Z \tag{45}$$

is the PGFL of $f_{k+1}(Z \mid X)$. Using (22) it can be shown that the PGFL of $f_{k+1|k+1}(X \mid Z^{(k+1)})$ is

$$G_{k+1|k+1}[h] = \frac{\frac{\delta F}{\delta Z_{k+1}}[0,h]}{\frac{\delta F}{\delta Z_{k+1}}[0,1]} \tag{46}$$

where the derivatives of $F[g,h]$ are taken with respect to g. Given the multitarget measurement model described at the beginning of Section IIG3, one derives a closed-form formula for $f_{k+1}(Z \mid X)$ and then for $F[g,h]$ [22, pp. 1173–1174]:

$$F[g,h] = e^{\lambda c[g]-\lambda} \cdot G_{k+1|k}[h(q_D + p_D p_g)] \tag{47}$$

where $q_D(\mathbf{x}) \triangleq 1 - p_D(\mathbf{x})$ and where

$$c[g] \triangleq \int g(\mathbf{z}) \cdot c(\mathbf{z})d\mathbf{z} \tag{48}$$

$$p_g(\mathbf{x}) \triangleq \int g(\mathbf{z}) \cdot f_{k+1}(\mathbf{z} \mid \mathbf{x})d\mathbf{z}. \tag{49}$$

To get a closed-form formula for the PHD corrector equation we must assume that $G_{k+1|k}[h]$ is Poisson, i.e., that $G_{k+1|k}[h] = e^{\mu s[h]-\mu}$ where $s[h] = \int h(\mathbf{x}) \cdot s(\mathbf{x})d\mathbf{x}$ and where $N_{k+1|k} = \mu$ and $D_{k+1|k}(\mathbf{x}) = \mu s(\mathbf{x})$. In this case (47) simplifies to

$$F[g,h] = \exp(\lambda c[g] - \lambda + \mu s[h(q_D + p_D p_g)] - \mu). \tag{50}$$

Using the product rule and chain rule for functional derivatives, we derive closed-form formulas for the numerator and denominator of (46). This yields a closed-form formula for $G_{k+1|k+1}[h]$. The data-updated PHD $D_{k+1|k+1}(\mathbf{x} \mid Z^{(k)})$ is derived in closed form using functional derivatives and (23):

$$D_{k+1|k+1}(\mathbf{x} \mid Z^{(k+1)}) = \frac{\delta G_{k+1|k+1}}{\delta \mathbf{x}}[1] \tag{51}$$

$$= \frac{\frac{\delta F}{\delta Z_{k+1}\delta \mathbf{x}}[0,1]}{\frac{\delta F}{\delta Z_{k+1}}[0,1]}. \tag{52}$$

Our derivation of the corrector step for the CPHD filter will employ identical reasoning, except that $e^{\lambda c[g]-\lambda}$ and $e^{\mu s[h]-\mu}$ will be replaced by IID cluster processes $C(c[g])$ and $G(s[h])$.

III. THE CPHD FILTER

In this section we derive equations for the predictor step (Section IIIA), corrector step (Section IIIB), and estimator step (Section IIIC) of the CPHD filter. Computational issues are addressed in Section IIID.

A. CPHD Filter Predictor

The CPHD filter presumes the following multitarget motion model: 1) target motions are statistically independent; 2) targets can disappear from the scene with state-dependent probability $p_S(\mathbf{x})$; and 3) new targets can appear in the scene independently of existing targets. Spawning of targets by other targets cannot be modeled.

From time-step k we have in hand the PHD $D_{k|k}(\mathbf{x} \mid Z^{(k)})$, the expected number of targets $N_{k|k}$, the cardinality distribution $p_{k|k}(n \mid Z^{(k)})$, and the PGF $G_{k|k}(x \mid Z^{(k)})$. We are to specify formulas for $D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$, $N_{k+1|k}$, $p_{k+1|k}(n \mid Z^{(k)})$, and

$G_{k+1|k}(x \mid Z^{(k)})$. Abbreviate $D_{k+1|k}(\mathbf{x}) = D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$, $p_{k+1|k}(n) = p_{k+1|k}(n \mid Z^{(k)})$, $G_{k+1|k}(x) = G_{k+1|k}(x \mid Z^{(k)})$, $D_{k|k}(\mathbf{x}) = D_{k|k}(\mathbf{x} \mid Z^{(k)})$, $p(n) = p_{k|k}(n \mid Z^{(k)})$, and $G(x) = G_{k|k}(x \mid Z^{(k)})$. Further assume that:

1) Motion of individual targets: $f_{k+1|k}(\mathbf{x} \mid \mathbf{x}')$ is the single-target Markov transition density.

2) Disappearance of existing targets: $p_{S,k+1|k}(\mathbf{x})$ is the probability that any target with state $\mathbf{x}$ at time-step k will survive in time-step $k+1$, and is hereafter abbreviated as $p_S(\mathbf{x})$.

3) Appearance of completely new targets: $b_{k+1|k}(X)$ is the likelihood that new targets with state-set X will enter the scene at time-step $k+1$, its PHD is denoted as $b(\mathbf{x})$, and the PGF of its cardinality distribution $p_B(n)$ as $B(x)$.

Closed-form formulas for the predicted PGF require the following simplifying assumption:

4) IID cluster process multitarget prior: The PGF of $f_{k|k}(X \mid Z^{(k)})$ has the form $G[h] = G(s[h])$ where $G(x) = G_{k|k}(x)$ is the PGF of the cardinality distribution of $f_{k|k}(X \mid Z^{(k)})$; where $s(\mathbf{x}) = N_{k|k}^{-1} \cdot D_{k|k}(\mathbf{x})$; and where $s[h] = \int h(\mathbf{x}) \cdot s(\mathbf{x}) d\mathbf{x}$.

Then:

THEOREM 1 (CPHD Filter Predictor) *For all $x \in [0,1]$ and all state-vectors $\mathbf{x}$,*

$$G_{k+1|k}(x) \cong B(x) \cdot G(1 - s[p_S] + s[p_S] \cdot x) \tag{53}$$

$$D_{k+1|k}(\mathbf{x}) = b(\mathbf{x}) + \int p_S(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot D_{k|k}(\mathbf{x}') d\mathbf{x}' \tag{54}$$

$$N_{k+1|k} = B_{k+1|k} + S_{k+1|k}. \tag{55}$$

Here $B_{k+1|k} = \int b(\mathbf{x}) d\mathbf{x}$ is the expected number of new targets and $S_{k+1|k}$ the expected number of surviving targets. Furthermore, the cardinality distribution corresponding to $G_{k+1|k}(x)$ is

$$p_{k+1|k}(n) \cong \sum_{i=0}^{n} p_B(n-i) \cdot \frac{1}{i!} \cdot G^{(i)}(1 - s[p_S]) \cdot s[p_S]^i \tag{56}$$

for all nonnegative integers n.

Equation (54) for the predicted PHD is exact since the same was true for the conventional PHD filter. Only (53) and (56) require proof, which may be found in Section IVA.

B. CPHD Filter Single-Sensor Corrector

The CPHD filter presumes the following multitarget measurement model: 1) a single target with state $\mathbf{x}$ generates, with probability $p_D(\mathbf{x})$, at most one observation; 2) any observation is generated by a single target; and 3) the false alarm process is an IID cluster process.

From the predictor step we have in hand the predicted PHD $D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$, the predicted expected number of targets $N_{k+1|k}$, the predicted PGF $G_{k+1|k}(x \mid Z^{(k)})$, and the predicted cardinality distribution $p_{k+1|k}(n \mid Z^{(k)})$. At time-step $k+1$ we collect a new observation-set $Z_{k+1} = \{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$ with m elements. We are to specify formulas for $D_{k+1|k+1}(\mathbf{x} \mid Z^{(k+1)})$, $N_{k+1|k+1}$, $G_{k+1|k+1}(x \mid Z^{(k+1)})$, and $p_{k+1|k+1}(n \mid Z^{(k+1)})$. We abbreviate $D_{k+1|k}(\mathbf{x}) = D_{k+1|k}(\mathbf{x} \mid Z^{(k)})$, $G[h] = G_{k+1|k}[h \mid Z^{(k)}]$, $G(x) = G_{k+1|k}(x \mid Z^{(k)})$, $p(n) = p_{k+1|k}(n \mid Z^{(k)})$, $D_{k+1|k+1}(\mathbf{x}) = D_{k+1|k+1}(\mathbf{x} \mid Z^{(k+1)})$, $p_{k+1|k+1}(n) = p_{k+1|k+1}(n \mid Z^{(k+1)})$, $G_{k+1|k+1}(x) = G_{k+1|k+1}(x \mid Z^{(k+1)})$, and $G_{k+1|k+1}[h] = G_{k+1|k+1}[h \mid Z^{(k+1)}]$.

Further,

1) Single-target measurement generation: $L_{k+1,\mathbf{z}}(\mathbf{x}) = f_{k+1}(\mathbf{z} \mid \mathbf{x})$ is the sensor likelihood function, hereafter abbreviated as $L_{\mathbf{z}}(\mathbf{x})$.

2) Probability of detection: $p_{D,k+1}(\mathbf{x})$ is the probability that an observation will be collected at time-step $k+1$ from a target with state $\mathbf{x}$. Hereafter we abbreviate $p_D(\mathbf{x}) = p_{D,k+1}(\mathbf{x})$ and $q_D(\mathbf{x}) = 1 - p_D(\mathbf{x})$.

3) IID cluster process false alarms: at time-step $k+1$ the sensor collects false alarms whose spatial distribution is given by the probability density $c_{k+1}(\mathbf{z})$ and whose cardinality distribution is given by $\kappa_{k+1}(m)$, hereafter abbreviated as $c(\mathbf{z})$ and $\kappa(m)$. The PGF of $\kappa_{k+1}(m)$ is $C_{k+1}(z)$, hereafter abbreviated as $C(z)$.

As was the case with the conventional PHD filter, we cannot derive closed-form formulas for the corrector step without making an additional simplifying assumption:

4) IID cluster process multitarget prior: the predicted multitarget distribution $f_{k+1|k}(X \mid Z^{(k)})$ is approximately an IID cluster process with PGFL $G[h] = G(s[h])$ where $G(x)$ is the PGF of the predicted cardinality distribution $p(n) = p_{k+1|k}(n)$; where $s[h] \triangleq \int h(\mathbf{x}) \cdot s(\mathbf{x}) d\mathbf{x}$; where $s(\mathbf{x}) \triangleq N_{k+1|k}^{-1} \cdot D_{k+1|k}(\mathbf{x})$; and where

$$N_{k+1|k} = \int D_{k+1|k}(\mathbf{x}) d\mathbf{x} = G^{(1)}(1). \tag{57}$$

The last equation results from

$$D_{k+1|k}(\mathbf{x}) = \left[\frac{\delta}{\delta \mathbf{x}} G(s[h]) \right]_{h=1} \tag{58}$$

$$= \left[G^{(1)}(s[h]) \cdot \frac{\delta}{\delta \mathbf{x}} s[h] \right]_{h=1} \tag{59}$$

$$= G^{(1)}(1) \cdot s(\mathbf{x}) \tag{60}$$

and thus $N_{k+1|k} = \int D_{k+1|k}(\mathbf{x})d\mathbf{x} = G^{(1)}(1)$. The ith derivatives of $G(x)$ and of $C(z)$ are $G^{(i)}(x)$ and $C^{(i)}(z)$, respectively.

Given this we have the following theorem.

THEOREM 2 (CPHD Filter Corrector) *The corrector step for the CPHD filter is given by the following approximate equalities. For all $x \in [0,1]$ and all state-vectors $\mathbf{x}$,*

$$G_{k+1|k+1}(x) \cong \frac{\sum_{j=0}^{m} x^j \cdot C^{(m-j)}(0) \cdot \hat{G}^{(j)}(xs[q_D]) \cdot \sigma_j(Z_{k+1})}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})} \tag{61}$$

$$D_{k+1|k+1}(\mathbf{x}) \cong L_{Z_{k+1}}(\mathbf{x}) \cdot D_{k+1|k}(\mathbf{x}) \tag{62}$$

where for all $\mathbf{x}$,

$$L_Z(\mathbf{x}) \triangleq \frac{\sum_{j=0}^{m} C^{(m-j)}(0) \cdot \hat{G}^{(j+1)}(s[q_D]) \cdot \sigma_j(Z)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z)} \cdot (1 - p_D(\mathbf{x})) + p_D(\mathbf{x}) \cdot \sum_{\mathbf{z}\in Z} \frac{L_{\mathbf{z}}(\mathbf{x})}{c(\mathbf{z})} \cdot \frac{\sum_{j=0}^{m-1} C^{(m-j-1)}(0) \cdot \hat{G}^{(j+1)}(s[q_D]) \cdot \sigma_j(Z - \{\mathbf{z}\})}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z)} \tag{63}$$

and where for any measurement-set $Z = \{\mathbf{z}_1,\ldots,\mathbf{z}_m\}$ with $|Z| = m$ and for all $x \in [0,1]$,

$$\sigma_i(Z) \triangleq \sigma_{m,i}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \tag{64}$$

$$\hat{G}^{(i)}(x) \triangleq \frac{G^{(i)}(x)}{G^{(1)}(1)^i} \tag{65}$$

where $\sigma_{m,i}(y_1,\ldots,y_m)$ is the elementary symmetric function of degree i in $y_1,\ldots,y_m$.

The proof can be found in Section IVB. Recall that $\sigma_{m,i}(y_1,\ldots,y_m)$ is defined as

$$\sigma_{m,i}(y_1,\ldots,y_m) = \sum_{0\le j_1<\cdots<j_i\le m} y_{j_1}\cdots y_{j_i} \tag{66}$$

$$= \sum_{S\subseteq U,|S|=i} \prod_{j\in S} y_j. \tag{67}$$

Thus $\sigma_{m,1}(y_1,\ldots,y_m) = y_1 + \cdots + y_m$, $\sigma_{m,m}(y_1,\ldots,y_m) = y_1\cdots y_m$, and by convention $\sigma_{m,0}(y_1,\ldots,y_m) = 1$ identically. Note that $\sigma_{m,i}(ay_1,\ldots,ay_m) = a^i \cdot \sigma_{m,i}(y_1,\ldots,y_m)$. Also, note that

$$\prod_{j=1}^{m}(1 + y_j) = \sum_{i=0}^{m} \sigma_{m,i}(y_1,\ldots,y_m). \tag{68}$$

Caution: The notation $\hat{G}^{(i)}(x)$ requires cautious handling. Note that

$$\frac{d^j}{dx^j}\hat{G}^{(i)}(x) = \hat{G}^{(i+j)}(x) \cdot G^{(1)}(1)^j \tag{69}$$

and not

$$\frac{d^j}{dx^j}\hat{G}^{(i)}(x) = \hat{G}^{(i+j)}(x). \tag{70}$$

For this reason we write

$$\hat{G}^{(i)(j)}(x) \triangleq \frac{d^j}{dx^j}\hat{G}^{(i)}(x). \tag{71}$$

Equations (61) and (62) are approximate in the sense that they depend on the assumption that $f_{k+1|k}(X \mid Z^{(k)})$ is approximately an IID cluster process. It is difficult to predict on an a priori basis conditions under which this assumption will be valid. For example, the Poisson assumption which underlies the corrector equation for the PHD filter implicitly depends on an assumption of "high SNR." Nevertheless, at least in basic simulations the PHD filter has proved to be effective in scenarios that, to the naked eye, seem to have rather small SNRs. The actual meaning of the IID cluster process assumption will have to be determined by experiment.

Closed-form formulas for the derivatives $G^{(i)}_{k+1|k+1}(x)$ and the cardinality distribution $p_{k+1|k+1}(n \mid Z^{(k)})$ are given in (89) and (90). The expected number of targets is

$$N_{k+1|k+1} = G^{(1)}_{k+1|k+1}(1) \tag{72}$$

$$\cong \frac{\sum_{j=0}^{m} C^{(m-j)}(0) \cdot \alpha_j \cdot \sigma_j(Z_{k+1})}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})} \tag{73}$$

where

$$\alpha_j \triangleq j \cdot \hat{G}^{(j)}(s[q_D]) + \hat{G}^{(j+1)}(s[q_D]) \cdot s[q_D] \cdot G^{(1)}(1) \tag{74}$$

$$= j \cdot \hat{G}^{(j)}(s[q_D]) + \hat{G}^{(j+1)}(s[q_D]) \cdot D_{k+1|k}[q_D]. \tag{75}$$

The following shows that (61)–(65) generalize the PHD filter corrector equations introduced in [22].

COROLLARY 1 (PHD Filter is Special Case of CPHD Filter) *Suppose that $C(z) = e^{\lambda z-\lambda}$ and $G(x) = e^{\mu x-\mu}$ are Poisson with $\mu = G^{(1)}(1)$. Then the CPHD corrector equations,* (61)–(65)*, reduce to the usual PHD filter corrector equations* (37) *and* (38).

The proof can be found in Section IVC.

The following describes a special case in which the CPHD corrector equations become exact rather than approximate. Suppose that we know a priori that there is at most one target in the scene. Then there is a $0 \leq \omega \leq 1$ and a probability density $f(\mathbf{x})$ such that

$$f_{k+1|k}(X \mid Z^{(k)}) = \begin{cases} 1-\omega & \text{if} \quad X = \emptyset \\ \omega \cdot f(\mathbf{x}) & \text{if} \quad X = \{\mathbf{x}\} \\ 0 & \text{if} \quad |X| \geq 2 \end{cases}. \tag{76}$$

It follows that that the PGFL $G[h] = G_{k+1|k}[h]$ of $f_{k+1|k}(X \mid Z^{(k)})$ is

$$G[h] = \int h^X \cdot f_{k+1|k}(X \mid Z^{(k)})\delta X \tag{77}$$

$$= f_{k+1|k}(X \mid Z^{(k)}) + \int h(\mathbf{x}) \cdot f_{k+1|k}(\{\mathbf{x}\} \mid Z^{(k)}) d\mathbf{x} \tag{78}$$

$$= 1 - \omega + \omega \cdot f[h] \tag{79}$$

where $f[h] = \int h(\mathbf{x}) \cdot f_{k+1|k}(\mathbf{x}) d\mathbf{x}$. The PGF of the cardinality distribution of $f_{k+1|k}(X \mid Z^{(k)})$ is, therefore, $G(x) = G[x] = 1 - \omega + \omega x$. Thus $G[h] = G(f[h])$ and $N_{k+1|k} = G^{(1)}(1) = \omega$. In this case the assumption that $G[h]$ is an IID cluster process is exact and we get Corollary 2.

COROLLARY 2 (CPHD and IPDA Filters) *Suppose that* $G(x) = 1 - \omega + \omega x$. *Then* (61)–(65) *reduce to*

$$G_{k+1|k+1}(x) = 1 - \omega_{k+1|k+1} + \omega_{k+1|k+1} \cdot x \tag{80}$$

$$D_{k+1|k+1}(\mathbf{x}) = L_{Z_{k+1}}(\mathbf{x}) \cdot D_{k+1|k}(\mathbf{x}) \tag{81}$$

$$N_{k+1|k+1} = \omega_{k+1|k+1} \tag{82}$$

where

$$\omega_{k+1|k+1} = \frac{\left(\begin{array}{c} C^{(m)}(0) \cdot \omega \cdot (1 - s[p_D]) \\ + C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \dfrac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array} \right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega s[p_D]) \\ + C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \dfrac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array} \right)} \tag{83}$$

and

$$L_Z(\mathbf{x}) = \frac{C^{(m)}(0)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega s[p_D]) \\ + C^{(m-1)}(0) \cdot \sum_{\mathbf{w} \in Z} \dfrac{D_{k+1|k}[p_D L_{\mathbf{w}}]}{c(\mathbf{w})} \end{array} \right)} \cdot (1 - p_D(\mathbf{x})) + p_D(\mathbf{x}) \cdot \frac{C^{(m-1)}(0) \cdot \left(\sum_{\mathbf{z} \in Z} \dfrac{L_{\mathbf{z}}(\mathbf{x})}{c(\mathbf{z})} \right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega s[p_D]) \\ + C^{(m-1)}(0) \cdot \sum_{\mathbf{w} \in Z} \dfrac{D_{k+1|k}[p_D L_{\mathbf{w}}]}{c(\mathbf{w})} \end{array} \right)}. \tag{84}$$

Furthermore,

$$p_{k+1|k+1}(0) = 1 - \omega_{k+1|k+1} \tag{85}$$

$$p_{k+1|k+1}(1) = \omega_{k+1|k+1}. \tag{86}$$

Thus $G_{k+1|k+1}[h] = G_{k+1|k+1}(f[h])$ where $D_{k+1|k+1}(\mathbf{x}) = N_{k+1|k+1} \cdot f(\mathbf{x})$. The proof may be found in Section IVD. Equations (80)–(86) generalize a previously known single-target detect-and-track filter, the IPDA filter [3].

In Corollary 2, suppose that 1) probability of detection $p_D(\mathbf{x}) = p_D$ is constant, and 2) no observations are collected ($Z_{k+1} = \emptyset$). Then (82) and (83) reduce to

$$N_{k+1|k+1} = \frac{(1 - p_D) \cdot \omega}{1 - \omega \cdot p_D} \tag{87}$$

which is equation (1.1) of Erdinc et al. We conclude: The CPHD filter is general enough to include the higher order information identified by Erdinc et al. as necessary for increasing the performance of the PHD filter.

C. CPHD Filter State Estimator

We noted in the Introduction that the PHD's EAP estimate $N_{k|k}$ of the posterior expected number of targets $\bar{n}$ can be both unstable and inaccurate under lower SNR conditions since $\bar{n}$ itself will be unstable. We further noted that the MAP estimate

$$\hat{n}_{k|k} \triangleq \arg\sup_n p_{k|k}(n \mid Z^{(k)}) \tag{88}$$

should be more stable and accurate. Its computation requires a closed-form formula for $p_{k|k}(n \mid Z^{(k)})$. This can be derived from (11).

THEOREM 3 (CPHD Estimator) *Under the conditions just stated,*

$$G^{(n)}_{k+1|k+1}(x) = \frac{\left(\begin{array}{c} \sum_{i=0}^{m} \sum_{j=0}^{n} C_{n,j} \cdot C^{(m-i)}(0) \cdot \sigma_i(Z_{k+1}) \\ \cdot \dfrac{i!}{(i-j)!} \cdot x^{i-j} \cdot \hat{G}^{(i)(n-j)}(x s[q_D]) \cdot s[q_D]^{n-j} \end{array} \right)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})} \tag{89}$$

and

$$p_{k+1|k+1}(n) = \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot \sigma_j(Z_{k+1}) \\ \cdot \dfrac{1}{(n-j)!} \cdot \hat{G}^{(j)(n-j)}(0) \cdot s[q_D]^{n-j} \end{array} \right)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})}. \tag{90}$$

The proof may be found in Section IVE. Note that $i! = \Gamma(i+1) = \pm\infty$ for all integers $i < 0$ where $\Gamma(x)$ is the gamma function, so that any terms in the summation in the numerators of (89) and (90) with $i > n$ vanish. The notation $\hat{G}^{(j)(i)}$ was defined in (71).

D. Computability of the CPHD Filter

It is evident that the CPHD filter will be more computationally intensive than the PHD filter. But how much more? The cardinality distributions $p_{k|k}(n \mid Z^{(k)})$ and $p_{k|k}(n \mid Z^{(k)})$ can be nonvanishing for all $n \geq 0$. Consequently, the CPHD filter will be inherently computationally intractable in the event that these distributions have heavy tails. Suppose on the other hand that these distributions vanish, at least approximately, for all n larger than some largest value ν. From (53) we see that this is the same thing as saying that their PGFs are polynomials of degrees not exceeding ν. What is the computability of the CPHD filter in this case?

Consider the corrector step first. Abbreviate $G(x) = G_{k+1|k}(x)$ and suppose that $\deg G(x) \leq \nu$. Then $\deg G^{(i)}(x) \leq \nu - i$. Examining (61) we find that

$$\deg G_{k+1|k+1}(x) \leq \deg \sum_{i=0}^{m} x^i \cdot C^{(m-i)}(0) \cdot \hat{G}^{(i)}(xs[q_D]) \cdot \sigma_i(Z_{k+1}) \tag{91}$$

$$\leq \max_i \{\deg x^i + \deg \hat{G}^{(i)}(xs[q_D])\} \tag{92}$$

$$\leq \max_i \{i + \nu - i\} = \nu. \tag{93}$$

Consequently, the corrector step will not increase computational requirements due to an increase in the maximum possible number of targets.

As for the predictor step, abbreviate $G(x) = G_{k|k}(x)$. Then from (53) we have

$$\deg G_{k+1|k}(x) = \deg\{B(x) \cdot G(1 - s[p_S] + s[p_S]x])\} \tag{94}$$

$$= \deg B(x) + \deg G(1 - s[p_S] + s[p_S]x]) \tag{95}$$

$$\leq \nu_B + \nu \tag{96}$$

where v_B is the maximum number of new targets. This will tend to increase $\deg G_{k+1|k}(x)$. Thus the computability of the CPHD filter is partially limited by what model of target appearance we adopt.

Beyond these contributing effects, the primary source of computational complexity will be the sums of the elementary symmetric functions $\sigma_1(Z), \ldots, \sigma_m(Z)$ in the corrector step. These sums are not as combinatorially daunting as they might at first seem. The reason is that $\sigma_{m,i}(x_1, \ldots, x_m)$ for $i = 1, \ldots, m$ can be computed with order m^2 complexity using the following double recursion on $j = 1, \ldots, m$ and $i = 1, \ldots, j$ [18, p. 40]:

$$\sigma_{j,1}(x_1, \ldots, x_m) = x_1 + \cdots + x_j \tag{97}$$

$$\sigma_{j,i}(x_1, \ldots, x_m) = \sigma_{j-1,i}(x_1, \ldots, x_{m-1}) + x_j \cdot \sigma_{j-1,i-1}(x_1, \ldots, x_{m-1}). \tag{98}$$

Consequently, taken as a whole the CPHD filter will have computational complexity $O(n \cdot m^3)$. Vo, Vo, and Cantoni [51] have shown that this can be somewhat reduced to $O(n \cdot m^2 \cdot \log^2 m)$.

IV. MATHEMATICAL PROOFS

A. Proof of CPHD Predictor

From (40) we know that the PGFL $G_{k+1|k}[h]$ of $f_{k+1|k}(X \mid Z^{(k)})$ may be expressed in terms of the PGFL $G[h] = G_{k|k}[h]$ of $f_{k|k}(X \mid Z^{(k)})$ as

$$G_{k+1|k}[h] = e_h \cdot G[(1 - p_S + p_S p_h) \cdot b_h] \tag{99}$$

where p_h, b_h, e_h are as in (40)–(43). Since we have assumed that no targets spawn other targets ($b_h(\mathbf{x}') = 1$ identically) and that $G[h] = G(s[h])$ this becomes

$$G_{k+1|k}[h] = e_h \cdot G(s[1 - p_S + p_S p_h]). \tag{100}$$

By (17) the PGF $G_{k+1|k}(x)$ of the cardinality distribution $p_{k+1|k}(n)$ of $f_{k+1|k}(X \mid Z^{(k)})$ is

$$G_{k+1|k}(x) = G_{k+1|k}[x] = e_x \cdot G(s[1 - p_S + p_S p_x]) \tag{101}$$

where

$$p_x(\mathbf{x}') = x \cdot \int f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') d\mathbf{x} = x \tag{102}$$

$$e_x = \int x^X \cdot b_{k+1|k}(X) \delta X \tag{103}$$

$$= \sum_{n=0}^{\infty} x^n \cdot \int_{|X|=n} b_{k+1|k}(X) \delta X \tag{104}$$

$$= \sum_{n=0}^{\infty} x^n \cdot b_{k+1|k}(n) = B(x). \tag{105}$$

Thus

$$G_{k+1|k}(x) = B(x) \cdot G(s[1 - p_S + x p_S]). \tag{106}$$

To determine $p_{k+1|k}(n)$ first note from the chain rule and general product rule for ordinary derivatives, (14), that

$$G_{k+1|k}^{(n)}(x) = \sum_{i=0}^{n} C_{n,i} \cdot \frac{d^{n-i}}{dx^{n-i}} B(x) \cdot \frac{d^i}{dx^i} G(s[1 - p_S + p_S x]) \tag{107}$$

$$= \sum_{i=0}^{n} C_{n,i} \cdot B^{(n-i)}(x) \cdot G^{(i)}(s[1 - p_S + p_S x]) \cdot s[p_S]^i. \tag{108}$$

Consequently, from (11)

$$p_{k+1|k}(n) = \frac{1}{n!} G^{(n)}_{k+1|k}(0) \tag{109}$$

$$= \frac{1}{n!} \sum_{i=0}^{n} C_{n,i} \cdot B^{(n-i)}(0) \cdot G^{(i)}(s[1-p_S]) \cdot s[p_S]^i \tag{110}$$

$$= \sum_{i=0}^{n} \frac{1}{(n-i)!} B^{(n-i)}(0) \cdot \frac{1}{i!} G^{(i)}(s[1-p_S]) \cdot s[p_S]^i \tag{111}$$

$$= \sum_{i=0}^{n} p_B(n-i) \cdot \frac{1}{i!} G^{(i)}(s[1-p_S]) \cdot s[p_S]^i \tag{112}$$

where $p_B(j) = (1/j!) b^{(j)}(0)$ is the distribution of the number of new targets.

B. Proof of CPHD Corrector

We first derive the formula for the PGF $G_{k+1|k+1}(x)$ and then, following that, the formula for the PHD $D_{k+1|k+1}(\mathbf{x})$. From (46) we know that the PGFL of the posterior distribution $f_{k+1|k+1}(X \mid Z^{(k+1)})$ is

$$G_{k+1|k+1}[h] = \frac{\dfrac{\delta F}{\delta Z_{k+1}}[0,h]}{\dfrac{\delta F}{\delta Z_{k+1}}[0,1]}. \tag{113}$$

By (47) we can replace the Poisson false alarm process $e^{\lambda c[g]-\lambda}$ by the IID cluster false alarm process $C(c[g])$ to get

$$F[g,h] \stackrel{\Delta}{=} C(c[g]) \cdot G(s[h(q_D + p_D p_g)]). \tag{114}$$

Begin with the numerator of (113). First note that

$$\frac{\delta^i}{\delta \mathbf{z}_1 \cdots \delta \mathbf{z}_i} C(c[g]) = C^{(i)}(c[g]) \cdot c(\mathbf{z}_1) \cdots c(\mathbf{z}_i) \tag{115}$$

and that

$$\frac{\delta^i}{\delta \mathbf{z}_1 \cdots \delta \mathbf{z}_i} G(s[h(q_D + p_D p_g)]) = G^{(i)}(s[h(q_D + p_D p_g)]) \cdot s[h p_D L_{\mathbf{z}_1}] \cdots s[h p_D L_{\mathbf{z}_i}]. \tag{116}$$

The second equation follows from the linear functional rule for functional derivatives, (24),

$$\frac{\delta}{\delta \mathbf{z}} s[h(q_D + p_D p_g)] = s\left[h p_D \frac{\delta}{\delta \mathbf{z}} p_g\right] = s[h p_D L_{\mathbf{z}}] \tag{117}$$

since

$$\frac{\delta}{\delta \mathbf{z}} p_g(\mathbf{x}) = \frac{\delta}{\delta \mathbf{z}} \int g(\mathbf{y}) \cdot L_{\mathbf{y}}(\mathbf{x}) d\mathbf{y} \tag{118}$$

$$= \int \delta_{\mathbf{z}}(\mathbf{y}) \cdot L_{\mathbf{y}}(\mathbf{x}) d\mathbf{y} = L_{\mathbf{z}}(\mathbf{x}). \tag{119}$$

Using the general product rule for functional derivatives, (26), the functional derivative of $F[g,h]$ with respect to the first variable g with respect to a measurement-set Z is

$$\frac{\delta F}{\delta Z}[g,h] = \sum_{W \subseteq Z} \frac{\delta}{\delta (Z-W)} C(c[g]) \cdot \frac{\delta}{\delta W} G(s[h(q_D + p_D p_g)]) \tag{120}$$

$$= \sum_{W \subseteq Z} C^{(|Z-W|)}(c[g]) \cdot \left(\prod_{\mathbf{z} \in Z-W} c(\mathbf{z})\right) \cdot G^{(|W|)}(s[h(q_D + p_D p_g)]) \cdot \left(\prod_{\mathbf{z} \in W} s[h p_D L_{\mathbf{z}}]\right) \tag{121}$$

$$= \left(\prod_{\mathbf{z} \in Z} c(\mathbf{z})\right) \cdot \sum_{W \subseteq Z} C^{(|Z-W|)}(c[g]) \cdot G^{(|W|)}(s[h(q_D + p_D p_g)]) \cdot \left(\prod_{\mathbf{z} \in W} \frac{s[h p_D L_{\mathbf{z}}]}{c(\mathbf{z})}\right) \tag{122}$$

$$= \left(\prod_{\mathbf{z} \in Z} c(\mathbf{z})\right) \cdot \sum_{i=1}^{m} C^{(m-i)}(c[g]) \cdot G^{(i)}(s[h(q_D + p_D p_g)]) \cdot \sum_{W \subseteq Z, |W|=i} \left(\prod_{\mathbf{z} \in W} \frac{s[h p_D L_{\mathbf{z}}]}{c(\mathbf{z})}\right) \tag{123}$$

$$= \left(\prod_{\mathbf{z} \in Z} c(\mathbf{z})\right) \cdot \sum_{i=1}^{m} C^{(m-i)}(c[g]) \cdot G^{(i)}(s[h(q_D + p_D p_g)]) \cdot \sigma_{m,i}\left(\frac{s[h p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[h p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \tag{124}$$

where $\sigma_{m,i}(x_1,\ldots,x_m)$ is as defined in (66) and (67). Substituting $g = 0$ into (124) we get

$$\frac{\delta F}{\delta Z}[0,h] = \left(\prod_{\mathbf{z} \in Z} c(\mathbf{z})\right) \cdot \sum_{i=1}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[h q_D]) \cdot \sigma_{m,i}\left(\frac{s[h p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[h p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right). \tag{125}$$

Substituting $h = 1$ into (125), we get

$$\frac{\delta F}{\delta Z}[0,1] = \left(\prod_{\mathbf{z}\in Z} c(\mathbf{z})\right) \cdot \sum_{i=1}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right). \quad (126)$$

Substituting (125) and (126) into (113), we find that the posterior PGFL is

$$G_{k+1|k+1}[h] = \frac{\left(\begin{array}{c}\sum_{j=1}^{m} C^{(m-j)}(0) \cdot G^{(j)}(s[hq_D]) \\ \cdot \sigma_{m,j}\left(\frac{s[hp_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[hp_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{array}\right)}{\left(\begin{array}{c}\sum_{i=1}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{array}\right)}. \quad (127)$$

By (17) the PGF of the posterior cardinality distribution $p_{k+1|k+1}(n)$ is, as claimed,

$$G_{k+1|k+1}(x) = G_{k+1|k+1}[x] \quad (128)$$

$$= \frac{\left(\begin{array}{c}\sum_{j=1}^{m} x^j \cdot C^{(m-j)}(0) \cdot G^{(j)}(xs[q_D]) \\ \cdot \sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{array}\right)}{\left(\begin{array}{c}\sum_{i=1}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{array}\right)} \quad (129)$$

$$= \frac{\left(\begin{array}{c}\sum_{j=1}^{m} x^j \cdot C^{(m-j)}(0) \cdot \hat{G}^{(j)}(xs[q_D]) \\ \cdot \sigma_{m,j}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{array}\right)}{\left(\begin{array}{c}\sum_{i=1}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{array}\right)} \quad (130)$$

$$= \frac{\sum_{j=1}^{m} x^j \cdot C^{(m-j)}(0) \cdot \hat{G}^{(j)}(xs[q_D]) \cdot \sigma_j(Z)}{\sum_{i=1}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z)} \quad (131)$$

where $\hat{G}^{(i)}(x) = G^{(i)}(x)/G^{(1)}(1)^i$; where $D_{k+1|k}(\mathbf{x}) = G^{(1)}(1) \cdot s(\mathbf{x})$ by (60); and where $\sigma_j(Z)$ are defined as in (64).

Now turn to the posterior PHD $D_{k+1|k+1}(\mathbf{x})$. By (51) it is given by

$$D_{k+1|k+1}(\mathbf{x}) = \frac{\frac{\delta F}{\delta Z_{k+1}\delta \mathbf{x}}[0,1]}{\frac{\delta F}{\delta Z_{k+1}}[0,1]} \quad (132)$$

where $(\delta/\delta Z)F[g,h]$ is taken with respect to variable g and $(\delta/\delta \mathbf{x})F[g,h]$ with respect to variable h.

By the linear functional rule, (24), the product rule, (25), and the chain rule, (27),

$$\frac{\delta F}{\delta \mathbf{x}}[g,h] = C(c[g]) \cdot \frac{\delta}{\delta \mathbf{x}} G(s[h(q_D + p_D p_g)]) \quad (133)$$

$$= C(c[g]) \cdot G^{(1)}(s[h(q_D + p_D p_g)]) \cdot (q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x})) \cdot s(\mathbf{x}). \quad (134)$$

Thus from the general product rule for functional derivatives, (26),

$$\frac{\delta F}{\delta Z \delta \mathbf{x}}[g,h] = \sum_{W\subseteq Z} \frac{\delta}{\delta(Z-W)} C(c[g]) \cdot \frac{\delta}{\delta W \delta \mathbf{x}} G(s[h(q_D + p_D p_g)]) \quad (135)$$

and thus

$$= \sum_{W\subseteq Z} C^{(|Z-W|)}(c[g]) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot \left(\begin{array}{c}\sum_{V\subseteq W} \frac{\delta}{\delta(W-V)} G^{(1)}(s[h(q_D + p_D p_g)]) \\ \cdot \frac{\delta}{\delta V}(q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x}))\end{array}\right) \cdot s(\mathbf{x}). \quad (136)$$

Since $q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x})$ is an affine functional in g, $(\delta/\delta V)(q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x})) = 0$ unless $V = \emptyset$ or $V = \{\mathbf{z}\}$ for some $\mathbf{z}$. Thus we get

$$\frac{\delta F}{\delta Z \delta \mathbf{x}}[g,h] = \sum_{W\subseteq Z} C^{(|Z-W|)}(c[g]) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot \left(\begin{array}{c}\frac{\delta}{\delta W} G^{(1)}(s[h(q_D + p_D p_g)]) \\ \cdot (q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x})) \\ + \sum_{\mathbf{z}\in W} \frac{\delta}{\delta(W-\mathbf{z})} G^{(1)}(s[h(q_D + p_D p_g)]) \\ \cdot \frac{\delta}{\delta \mathbf{z}}(q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x}))\end{array}\right) \cdot s(\mathbf{x}) \quad (137)$$

and so

$$= \sum_{W\subseteq Z} C^{(|Z-W|)}(c[g]) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot \left(\begin{array}{c} G^{(1+|W|)}(s[h(q_D + p_D p_g)]) \\ \cdot \left(\prod_{\mathbf{z}\in W} s[hp_D L_{\mathbf{z}}]\right) \cdot (q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x})) \\ + \sum_{\mathbf{z}\in W} G^{(|W|)}(s[h(q_D + p_D p_g)]) \\ \cdot \left(\prod_{\mathbf{w}\in W, \mathbf{w}\neq \mathbf{z}} s[hp_D L_{\mathbf{w}}]\right) \cdot p_D(\mathbf{x}) L_{\mathbf{z}}(\mathbf{x})\end{array}\right) \cdot s(\mathbf{x}) \quad (138)$$

and so

$$= \sum_{W\subseteq Z} C^{(|Z-W|)}(c[g]) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot G^{(1+|W|)}(s[h(q_D + p_D p_g)]) \cdot \left(\prod_{\mathbf{z}\in W} s[hp_D L_{\mathbf{z}}]\right) \cdot (q_D(\mathbf{x}) + p_D(\mathbf{x}) p_g(\mathbf{x})) \cdot s(\mathbf{x}) + \sum_{W\subseteq Z} C^{(|Z-W|)}(c[g]) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot \left(\begin{array}{c} \sum_{\mathbf{z}\in W} G^{(|W|)}(s[h(q_D + p_D p_g)]) \\ \cdot \left(\prod_{\mathbf{w}\in W, \mathbf{w}\neq\mathbf{z}} s[hp_D L_{\mathbf{w}}]\right) \cdot p_D(\mathbf{x}) L_{\mathbf{z}}(\mathbf{x}) \end{array}\right) \cdot s(\mathbf{x}). \quad (139)$$

Setting $g = 0$ and $h = 1$,

$$\frac{\delta F}{\delta Z \delta \mathbf{x}}[0,1] = \sum_{W\subseteq Z} C^{(|Z-W|)}(0) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot G^{(1+|W|)}(s[q_D]) \cdot \left(\prod_{\mathbf{z}\in W} s[p_D L_{\mathbf{z}}]\right) \cdot q_D(\mathbf{x}) \cdot s(\mathbf{x}) + \sum_{W\subseteq Z} C^{(|Z-W|)}(0) \cdot \left(\prod_{\mathbf{z}\in Z-W} c(\mathbf{z})\right) \cdot \left(\sum_{\mathbf{z}\in W} G^{(|W|)}(s[q_D]) \cdot \left(\prod_{\mathbf{w}\in W, \mathbf{w}\neq\mathbf{z}} s[p_D L_{\mathbf{w}}]\right)\right) \cdot p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x}) \cdot s(\mathbf{x}) \quad (140)$$

and so

$$= \left(\prod_{\mathbf{z}\in Z} c(\mathbf{z})\right) \cdot \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i+1)}(s[q_D]) \cdot \sum_{W\subseteq Z, |W|=i} \left(\prod_{\mathbf{w}\in W} \frac{s[p_D L_{\mathbf{w}}]}{c(\mathbf{w})}\right) \cdot q_D(\mathbf{x}) \cdot s(\mathbf{x}) + \left(\prod_{\mathbf{z}\in Z} c(\mathbf{z})\right) \cdot \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \cdot \sum_{W\subseteq Z, |W|=i} \left(\sum_{\mathbf{z}\in Z} \frac{L_{\mathbf{z}}(\mathbf{x}) \cdot \mathbf{1}_W(\mathbf{z})}{s[p_D L_{\mathbf{z}}]}\right) \cdot \left(\prod_{\mathbf{w}\in W} \frac{s[p_D L_{\mathbf{w}}]}{c(\mathbf{w})}\right) \cdot p_D(\mathbf{x}) \cdot s(\mathbf{x})$$

and thus from (67)

$$= \left(\prod_{\mathbf{z}\in Z} c(\mathbf{z})\right) \cdot \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i+1)}(s[q_D]) \cdot \sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \cdot q_D(\mathbf{x}) \cdot s(\mathbf{x}) + \left(\prod_{\mathbf{z}\in Z} c(\mathbf{z})\right) \cdot \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \cdot \sum_{W\subseteq Z, |W|=i} \left(\sum_{\mathbf{z}\in Z} \frac{L_{\mathbf{z}}(\mathbf{x}) \cdot \mathbf{1}_W(\mathbf{z})}{s[p_D L_{\mathbf{z}}]}\right) \cdot \left(\prod_{\mathbf{w}\in W} \frac{s[p_D L_{\mathbf{w}}]}{c(\mathbf{w})}\right) \cdot p_D(\mathbf{x}) \cdot s(\mathbf{x}). \quad (141)$$

Substituting (141) and (126) into (132) and canceling the factor $\prod_{\mathbf{z}\in Z} c(\mathbf{z})$ in the numerator and denominator, we find that the posterior PHD is

$$D_{k+1|k+1}(\mathbf{x}) = \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot G^{(1+j)}(s[q_D]) \\ \cdot \sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \end{array}\right)}{\left(\begin{array}{c} \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \end{array}\right)} \cdot q_D(\mathbf{x}) \cdot s(\mathbf{x}) + \sum_{\mathbf{z}\in Z} \frac{p_D(\mathbf{x}) L_{\mathbf{z}}(\mathbf{x}) s(\mathbf{x})}{s[p_D L_{\mathbf{z}}]} \cdot \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot G^{(j)}(s[q_D]) \\ \cdot \sum_{W\subseteq Z, |W|=j} \mathbf{1}_W(\mathbf{z}) \cdot \left(\prod_{\mathbf{z}\in W} \frac{s[p_D L_{\mathbf{z}}]}{c(\mathbf{z})}\right) \end{array}\right)}{\left(\begin{array}{c} \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \end{array}\right)} \quad (142)$$

and so

$$= \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot G^{(1+j)}(s[q_D]) \\ \cdot \sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \end{array}\right)}{\left(\begin{array}{c} \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \end{array}\right)} \cdot q_D(\mathbf{x}) \cdot s(\mathbf{x}) + \sum_{\mathbf{z}\in Z} \frac{p_D(\mathbf{x}) L_{\mathbf{z}}(\mathbf{x}) s(\mathbf{x})}{c(\mathbf{z})} \cdot \frac{\left(\begin{array}{c} \sum_{j=1}^{m} C^{(m-j)}(0) \cdot G^{(j)}(s[q_D]) \\ \cdot \sum_{V\subseteq Z-\mathbf{z}, |V|=j-1} \prod_{\mathbf{w}\in V} \frac{s[p_D L_{\mathbf{w}}]}{c(\mathbf{w})} \end{array}\right)}{\left(\begin{array}{c} \sum_{i=0}^{m} C^{(m-i)}(0) \cdot G^{(i)}(s[q_D]) \\ \cdot \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right) \end{array}\right)} \quad (143)$$

or

$$D_{k+1|k+1}(\mathbf{x}) = \frac{\begin{pmatrix} \sum_{j=0}^{m} C^{(m-j)}(0)\cdot G^{(1+j)}(s[q_D]) \\ \cdot\sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} C^{(m-i)}(0)\cdot G^{(i)}(s[q_D]) \\ \cdot\sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \cdot q_D(\mathbf{x})\cdot s(\mathbf{x}) + \sum_{\mathbf{z}\in Z}\frac{p_D(\mathbf{x})L_{\mathbf{z}}(\mathbf{x})s(\mathbf{x})}{s[p_D L_{\mathbf{z}}]}\cdot\frac{s[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \cdot \frac{\begin{pmatrix} \sum_{j=0}^{m-1} C^{(m-j-1)}(0)\cdot G^{(j+1)}(s[q_D]) \\ \cdot\sum_{V\subseteq Z-\mathbf{z},|V|=j}\prod_{\mathbf{w}\in V}\frac{s[p_D L_{\mathbf{w}}]}{c(\mathbf{w})}\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} C^{(m-i)}(0)\cdot G^{(i)}(s[q_D]) \\ \cdot\sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \tag{144}$$

$$= \frac{\begin{pmatrix} \sum_{j=0}^{m} C^{(m-j)}(0)\cdot G^{(1+j)}(s[q_D]) \\ \cdot\sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} C^{(m-i)}(0)\cdot G^{(i)}(s[q_D]) \\ \cdot\sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \cdot q_D(\mathbf{x})\cdot s(\mathbf{x}) + \sum_{\mathbf{z}\in Z}\frac{p_D(\mathbf{x})L_{\mathbf{z}}(\mathbf{x})s(\mathbf{x})}{c(\mathbf{z})} \cdot \frac{\begin{pmatrix} \sum_{j=0}^{m-1} C^{(m-j-1)}(0)\cdot G^{(j+1)}(s[q_D]) \\ \cdot\sigma_{m-1,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\widehat{\frac{s[p_D L_{\mathbf{z}}]}{c(\mathbf{z})}},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} C^{(m-i)}(0)\cdot G^{(i)}(s[q_D]) \\ \cdot\sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \tag{145}$$

where $\sigma_{m-1,j}(x_1,\ldots,\widehat{x_i},\ldots,x_m)$ indicates that the ith variable is to be struck out of the list $x_1,\ldots,x_i,\ldots,x_m$. Now recall from (12) that the prior PHD is

$$D_{k+1|k}(\mathbf{x}) = \frac{\delta G}{\delta\mathbf{x}}[1] = G^{(1)}(1)\cdot s(\mathbf{x}). \tag{146}$$

Thus if we define $\hat{G}^{(i)}(x) \triangleq G^{(i)}(x)/G^{(1)}(1)^i$ we get

$$D_{k+1|k+1}(\mathbf{x}) = \frac{\begin{pmatrix} \sum_{j=0}^{m} C^{(m-j)}(0)\cdot \hat{G}^{(1+j)}(s[q_D]) \\ \cdot\sigma_{m,j}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} C^{(m-i)}(0)\cdot \hat{G}^{(i)}(s[q_D]) \\ \cdot\sigma_{m,i}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \cdot q_D(\mathbf{x})\cdot D_{k+1|k}(\mathbf{x}) + \sum_{\mathbf{z}\in Z}\frac{p_D(\mathbf{x})L_{\mathbf{z}}(\mathbf{x})D_{k+1|k}(\mathbf{x})}{c(\mathbf{z})} \cdot \frac{\begin{pmatrix} \sum_{j=0}^{m-1} C^{(m-j-1)}(0)\cdot \hat{G}^{(j+1)}(s[q_D]) \\ \cdot\sigma_{m-1,j}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\widehat{\frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})}},\ldots,\frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} C^{(m-i)}(0)\cdot \hat{G}^{(i)}(s[q_D]) \\ \cdot\sigma_{m,i}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \tag{147}$$

or

$$D_{k+1|k+1}(\mathbf{x}) = L_Z(\mathbf{x})\cdot D_{k+1|k}(\mathbf{x}) \tag{148}$$

where

$$L_Z(\mathbf{x}) = \frac{\sum_{j=0}^{m} C^{(m-j)}(0)\cdot\hat{G}^{(j+1)}(s[q_D])\cdot\sigma_j(Z)}{\sum_{i=0}^{m} C^{(m-i)}(0)\cdot\hat{G}^{(i)}(s[q_D])\cdot\sigma_i(Z)} \cdot (1-p_D(\mathbf{x})) + p_D(\mathbf{x})\cdot\sum_{\mathbf{z}\in Z}\frac{L_{\mathbf{z}}(\mathbf{x})}{c(\mathbf{z})} \cdot\frac{\sum_{j=0}^{m-1} C^{(m-j-1)}(0)\cdot\hat{G}^{(j+1)}(s[q_D])\cdot\sigma_j(Z-\{\mathbf{z}\})}{\sum_{i=0}^{m} C^{(m-i)}(0)\cdot\hat{G}^{(i)}(s[q_D])\cdot\sigma_i(Z)} \tag{149}$$

where for any $Z = \{\mathbf{z}_1,\ldots,\mathbf{z}_m\}$ with $|Z| = m$ and any $m \geq 1$,

$$\sigma_i(Z) \triangleq \sigma_{m,i}\left(\frac{D_{k+1|k}[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{D_{k+1|k}[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right). \tag{150}$$

This completes the proof.

C. Proof of Corollary 1

In (61)–(65) we are to assume that $C(z) = e^{\lambda z-\lambda}$ and $G(x) = e^{\mu x-\mu}$ where $\mu = N_{k+1|k}$ is the predicted expected number of targets and $\mu s(\mathbf{x}) = D_{k+1|k}(\mathbf{x})$ is the predicted PHD. Noting that $C^{(i)}(0) = \lambda^i\cdot e^{-\lambda}$ and $G^{(i)}(s[q_D]) = \mu^i\cdot e^{\mu s[p_D]-\mu}$ and substituting these quantities into (147), we get

$$D_{k+1|k+1}(\mathbf{x}) = \frac{\begin{pmatrix} \sum_{j=0}^{m} \lambda^{m-j}\mu^{j+1} \\ \cdot\sigma_{m,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} \lambda^{m-i}\mu^{i} \\ \cdot\sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \cdot q_D(\mathbf{x})\cdot s(\mathbf{x}) + \sum_{\mathbf{z}\in Z}\frac{p_D(\mathbf{x})\cdot L_{\mathbf{z}}(\mathbf{x})\cdot s(\mathbf{x})}{c(\mathbf{z})} \cdot \frac{\begin{pmatrix} \sum_{j=0}^{m-1} \lambda^{m-j-1}\mu^{j+1} \\ \cdot\sigma_{m-1,j}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\widehat{\frac{s[p_D L_{\mathbf{z}}]}{c(\mathbf{z})}},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}}{\begin{pmatrix} \sum_{i=0}^{m} \lambda^{m-i}\mu^{i} \\ \cdot\sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)},\ldots,\frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)\end{pmatrix}} \tag{151}$$

$$= \frac{\sum_{j=0}^{m} \sigma_{m,j}\left(\frac{\mu s[p_D L_{\mathbf{z}_1}]}{\lambda c(\mathbf{z}_1)}, \ldots, \frac{\mu s[p_D L_{\mathbf{z}_m}]}{\lambda c(\mathbf{z}_m)}\right)}{\sum_{i=0}^{m} \sigma_{m,i}\left(\frac{s[p_D L_{\mathbf{z}_1}]}{c(\mathbf{z}_1)}, \ldots, \frac{s[p_D L_{\mathbf{z}_m}]}{c(\mathbf{z}_m)}\right)}$$

$$\cdot\, q_D(\mathbf{x}) \cdot \mu s(\mathbf{x}) + \sum_{\mathbf{z} \in Z} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x}) \cdot \mu s(\mathbf{x})}{\lambda c(\mathbf{z})}$$

$$\cdot\, \frac{\sum_{j=0}^{m-1} \sigma_{m-1,j}\left(\begin{array}{c} \frac{\mu s[p_D L_{\mathbf{z}_1}]}{\lambda c(\mathbf{z}_1)}, \ldots, \widehat{\frac{\mu s[p_D L_{\mathbf{z}}]}{\lambda c(\mathbf{z})}}, \\ \ldots, \frac{\mu s[p_D L_{\mathbf{z}_m}]}{\lambda c(\mathbf{z}_m)} \end{array}\right)}{\sum_{i=0}^{m} \sigma_{m,i}\left(\frac{\mu s[p_D L_{\mathbf{z}_1}]}{\lambda c(\mathbf{z}_1)}, \ldots, \frac{\mu s[p_D L_{\mathbf{z}_m}]}{\lambda c(\mathbf{z}_m)}\right)}. \quad (153)$$

From the identity for elementary symmetric functions, (68), this becomes

$$D_{k+1|k+1}(\mathbf{x}) = q_D(\mathbf{x}) \cdot \mu s(\mathbf{x}) + \sum_{\mathbf{z} \in Z} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x}) \cdot \mu s(\mathbf{x})}{\lambda c(\mathbf{z})} \cdot \frac{1}{1 + \frac{\mu s[p_D L_{\mathbf{z}}]}{\lambda c(\mathbf{z})}} \quad (154)$$

$$= q_D(\mathbf{x}) \cdot \mu s(\mathbf{x}) + \sum_{\mathbf{z} \in Z} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x}) \cdot \mu s(\mathbf{x})}{\lambda c(\mathbf{z}) + \mu s[p_D L_{\mathbf{z}}]}. \quad (155)$$

Substituting $\mu s(\mathbf{x}) = D_{k+1|k}(\mathbf{x})$ and $q_D(\mathbf{x}) = 1 - p_D(\mathbf{x})$ yields the claimed result.

D. Proof of Corollary 2

From (53) we have

$$G_{k+1|k+1}(x) = \frac{\sum_{j=0}^{m} x^j \cdot C^{(m-j)}(0) \cdot \hat{G}^{(j)}(xs[q_D]) \cdot \sigma_j(Z_{k+1})}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})} \quad (156)$$

$$= \frac{\left(\begin{array}{c} C^{(m)}(0) \cdot G(xs[q_D]) \\ +x \cdot C^{(m-1)}(0) \cdot \hat{G}^{(1)}(xs[q_D]) \\ \cdot \sigma_1(Z_{k+1}) \end{array}\right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot G(s[q_D]) \\ +C^{(m-1)}(0) \cdot \hat{G}^{(1)}(s[q_D]) \\ \cdot \sigma_1(Z_{k+1}) \end{array}\right)} \quad (157)$$

$$= \frac{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega + \omega x s[q_D]) \\ +x \cdot C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega + \omega s[q_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)} \quad (158)$$

$$= \frac{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega) \\ +x \cdot \left(\begin{array}{c} C^{(m)}(0) \cdot \omega \cdot (1 - s[p_D]) \\ +C^{(m-1)}(0) \\ \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right) \end{array}\right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega s[p_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)} \quad (159)$$

$$= 1 - \omega_{k+1|k+1} + \omega_{k+1|k+1} \cdot x \quad (160)$$

where, as claimed,

$$\omega_{k+1|k+1} = \frac{\left(\begin{array}{c} C^{(m)}(0) \cdot \omega \cdot (1 - s[p_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega s[p_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)} \quad (161)$$

$$1 - \omega_{k+1|k+1} = \frac{C^{(m)}(0) \cdot (1 - \omega)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega s[p_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z_{k+1}} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)}. \quad (162)$$

As for the PHD, from (63) we get

$$L_Z(\mathbf{x}) = \frac{\sum_{j=0}^{m} C^{(m-j)}(0) \cdot \hat{G}^{(j+1)}(s[q_D]) \cdot \sigma_j(Z)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z)} \cdot (1 - p_D(\mathbf{x})) + p_D(\mathbf{x}) \cdot \sum_{\mathbf{z} \in Z} \frac{L_{\mathbf{z}}(\mathbf{x})}{c(\mathbf{z})} \cdot \frac{\left(\begin{array}{c} \sum_{j=0}^{m-1} C^{(m-j-1)}(0) \\ \cdot \hat{G}^{(j+1)}(s[q_D]) \cdot \sigma_j(Z - \{\mathbf{z}\}) \end{array}\right)}{\left(\begin{array}{c} \sum_{i=0}^{m} C^{(m-i)}(0) \\ \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z) \end{array}\right)} \quad (163)$$

$$= \frac{C^{(m)}(0)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega + \omega s[q_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{z} \in Z} \frac{D_{k+1|k}[p_D L_{\mathbf{z}}]}{c(\mathbf{z})} \end{array}\right)} \cdot (1 - p_D(\mathbf{x})) + p_D(\mathbf{x}) \cdot \frac{C^{(m-1)}(0) \cdot \left(\sum_{\mathbf{z} \in Z} \frac{L_{\mathbf{z}}(\mathbf{x})}{c(\mathbf{z})}\right)}{\left(\begin{array}{c} C^{(m)}(0) \cdot (1 - \omega + \omega s[q_D]) \\ +C^{(m-1)}(0) \cdot \sum_{\mathbf{w} \in Z} \frac{D_{k+1|k}[p_D L_{\mathbf{w}}]}{c(\mathbf{w})} \end{array}\right)} \quad (164)$$

as claimed.

E. Proof of CPHD Estimator

From (61)

$$G_{k+1|k+1}(x) \cong \frac{\sum_{j=0}^{m} x^j \cdot C^{(m-j)}(0) \cdot \hat{G}^{(j)}(xs[q_D]) \cdot \sigma_j(Z_{k+1})}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})}. \tag{165}$$

Using the general product rule

$$\frac{d^n(G_1 G_2)}{dx^n}(x) = \sum_{j=0}^{n} C_{n,j} \cdot \frac{d^{n-j}G_1}{dx^{n-j}}(x) \cdot \frac{d^j G_2}{dx^j}(x) \tag{166}$$

we get

$$G^{(n)}_{k+1|k+1}(x) = \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot \sigma_j(Z_{k+1}) \\ \cdot \sum_{e=0}^{n} C_{n,e} \cdot \left(\frac{d^e}{dx^e} x^j\right) \cdot \left(\frac{d^{n-e}}{dx^{n-e}} \hat{G}^{(j)}(xs[q_D])\right) \end{array}\right)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})}$$

$$= \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot \sigma_j(Z_{k+1}) \cdot \sum_{e=0}^{n} C_{n,e} \\ \cdot \frac{j!}{(j-e)!} \cdot x^{j-e} \cdot \hat{G}^{(j)(n-e)}(xs[q_D]) \cdot s[q_D]^{n-e} \end{array}\right)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})}. \tag{167}$$

After substitution of $x = 0$, only those terms in the numerator with $j = e$ survive and so we get, as claimed,

$$p_{k+1|k+1}(n) = \frac{1}{n!} G^{(n)}_{k+1|k+1}(0)$$

$$= \frac{1}{n!} \cdot \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot \sigma_j(Z_{k+1}) \\ \cdot C_{n,j} \cdot j! \cdot \hat{G}^{(j)(n-j)}(0) \cdot s[q_D]^{n-j} \end{array}\right)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})} \tag{168}$$

$$= \frac{\left(\begin{array}{c} \sum_{j=0}^{m} C^{(m-j)}(0) \cdot \sigma_j(Z_{k+1}) \\ \cdot \frac{1}{(n-j)!} \cdot \hat{G}^{(j)(n-j)}(0) \cdot s[q_D]^{n-j} \end{array}\right)}{\sum_{i=0}^{m} C^{(m-i)}(0) \cdot \hat{G}^{(i)}(s[q_D]) \cdot \sigma_i(Z_{k+1})}. \tag{169}$$

V. CONCLUSIONS

In a recent paper [16], Erdinc, Willett, and Bar-Shalom argued that the performance of the PHD filter could be improved if it could be generalized to include second-order information regarding target number. In this paper we derived closed-form predictor and corrector equations for the CPHD filter, which propagates not only the PHD but also the entire cardinality distribution (probability distribution on target number). We suggested that estimation of target number under lower-SNR conditions could be further improved by extracting MAP estimates rather than EAP estimates from the cardinality distribution.

Vo, Vo, and Cantoni have devised and successfully tested Gaussian-mixture implementations of the CPHD filter [50, 51, 53]. In one simulation, for example, five targets appear and disappear while observed by a linear-Gaussian sensor in a dense Poisson false alarm environment. The CPHD filter correctly detected all target births and deaths and successfully tracked the targets during the times they were present in the scene. Similar results were reported for a similar simulation involving a nonlinear (range-bearing) sensor and nonlinear target dynamics. In a third simulation, Gaussian-mixture implementations of the PHD and CPHD filters were compared in scenarios with up to ten targets appearing randomly with multiple track crossings. Both filters successfully identified target births and deaths, tracked the targets, and negotiated track crossings. As expected, for any individual sample path, the CPHD filter's estimates of instantaneous target number were far more accurate and stable (small variance) than those of the PHD filter.

Suggested future research: 1) implementation of the CPHD filter using sequential Monte Carlo techniques, 2) further testing of the CPHD filter in more complex scenarios, and 3) since the corrector step of the CPHD filter will become intractable when the number of measurements is too large, further approximations will be necessary. The author has reported an only partially successful attack on this problem [25].

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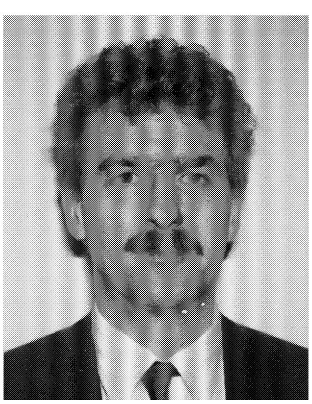

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