PMBM density
Multi-Object Tracking

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A MOTIVATION FOR CHANGING THE BIRTH MODEL

In MBM filters, an MB birth model is used:

- In the prediction, add $N_k^B$ birth components to each MB in the mixture
- If an object actually appeared and was detected, then in the update this detection can be associated to the birth component, and we can start to track the object.

But why add birth components before we know if there are any detections?

- Could the addition of new Bernoulli components, corresponding to the new potential objects, not be measurement-driven?
- Yes, it could, if we use a PPP birth model instead of a MB birth model.
With a Poisson birth, the Poisson Multi-Bernoulli Mixture (PMBM) density

\[ \mathcal{PMBM}_{k|k}(x_k) \]

is a multi-object conjugate prior to the standard point object transition density \( p(x_k|x_{k-1}) \) and measurement model \( p(z_k|x_k) \),

Prediction: 
\[ \mathcal{PMBM}_{k|k-1}(x_k) = \int p(x_k|x_{k-1}) \mathcal{PMBM}_{k-1|k-1}(x_{k-1}) \delta x_{k-1} \]

Update: 
\[ \mathcal{PMBM}_{k|k}(x_k) = \frac{p(z_k|x_k) \mathcal{PMBM}_{k|k-1}(x_k)}{\int p(z_k|x'_k) \mathcal{PMBM}_{k|k-1}(x'_k) \delta x'_k}. \]
Beyond multi-object conjugacy, why is the PMBM model useful for MOT?

Some uncertainties in MOT:

- Are there any objects? How many?
  - Detected objects: Bernoulli existence probabilities
  - Undetected objects: PPP intensity

- If so, what are their states?
  - Detected objects: Bernoulli state densities
  - Undetected objects: PPP intensity

- Data association? Captured by the MB mixture.

The PMBM density nicely captures the relevant uncertainties.
In the PMBM model, the set of objects $x_k$ at time $k$

- Union of detected objects and undetected objects $x_k = x_k^d \cup x_k^u$
  - Detected $x_k^d$: objects the sensors have detected at least once
  - Undetected $x_k^u$: objects that have never detected

We are doing tracking based on detections, how could we track undetected objects?!

- Representation of their possible existence.
- Actually included in many tracking algorithms.
  - MBM filter: any Bernoulli to which a detection has never been associated
- Here it is made explicit.
Autonomous car: Possibility of undetected objects in occluded areas
The Poisson Multi-Bernoulli Mixture density is defined as

\[
\mathcal{PMBM}_{k|k}(x_k) = \sum_{x_k^u \cup x_k^d = x_k} \mathcal{P}^u_{k|k}(x_k^u) \mathcal{MBM}^d_{k|k}(x_k^d)
\]

PPP density \( \mathcal{P}^u_{k|k}(\cdot) \) for undetected objects, typically with mixture intensity,

\[
\lambda^u_{k|k}(x_k) = \sum_{t=1}^{N^u_k} w^u_{k|k} p^u_{k|k}(x_k), \quad \left\{ \left( w^u_{k|k}, p^u_{k|k}(\cdot) \right) \right\}_{t=1}^{N^u_k}
\]

Multi-Bernoulli Mixture density \( \mathcal{MBM}^d_{k|k}(\cdot) \) for detected objects, with parameters

\[
\left\{ \left( \ell^h_{k|k}, \left\{ \left( r^i_{k|k}, p^i_{k|k}(\cdot) \right) \right\}_{i=1}^{N^h_k} \right) \right\}_{h=1}^{\mathcal{H}_k}
\]

PMBM density parameterised by the mixture intensity parameters, the MB log-weights, and the Bernoulli parameters of the MBs.
Example: PPP mixture intensity and Bernoulli state pdfs are Gaussian

Gaussian mixture intensity,

$$\lambda_{k|k}^u(x_k) = \sum_{t=1}^{N_{k|k}^u} w_{k|k}^{u,t} \mathcal{N}(x_k; \mu_{k|k}^{u,t}, P_{k|k}^{u,t})$$

Gaussian object densities,

$$p_{k|k}^{i,h_k}(x_k^{i,h_k}) = \mathcal{N}(x_k^{i,h_k}; \mu_{k|k}^{i,h_k}, P_{k|k}^{i,h_k})$$

PMBM density parameters

$$\left\{ \left( w_{k|k}^{u,t}, \mu_{k|k}^{u,t}, P_{k|k}^{u,t} \right) \right\}_{t=1}^{N_{k|k}^u}, \left\{ \left( \ell_{k|k}^{h_k}, \left\{ \left( r_{k|k}^{i,h_k}, \mu_{k|k}^{i,h_k}, P_{k|k}^{i,h_k} \right) \right\}_{i=1}^{N_{h_k}^k} \right) \right\}_{h_k=1}^{\mathcal{H}_k}$$
If we design an MOT algorithm for the PMBM density, we get a PMBM filter

**PMBM filter: pseudo-code**

For $k = 1, 2, \ldots, K$

- Prediction
- Update
- Reduction
- Estimation
PMBM prediction

Multi-Object Tracking

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**POISSON MULTI-BERNOULLI MIXTURE PREDICTION**

- **Posterior PMBM parameters**
- **Prediction**

\[
\mathcal{PMBM}_{k+1|k}(x_k) = \int p(x_{k+1}|x_k) \mathcal{PMBM}_{k|k}(x_k) \delta x_k
\]

with transition density \( p(x_{k+1}|x_k) \) with

- Probability of survival \( P_S^k(x_k) \)
- Transition density \( \pi_{k+1}(x_{k+1}|x_k) \)
- PPP birth model: \( \left\{ w_{k+1}^B, r_{k+1}^B \right\}^{N_{k+1}^B}_{i=1} \)

- **Predicted PMBM parameters**
PMBM prediction with PPP birth

- Undetected and detected objects can be predicted independently
- Predicted undetected parameters consist of union of
  - predicted parameters from previous time step, and
  - birth parameters
- Each MB can be predicted independently of the other MBs
- Number of parameters increases (we add PPP birth params. to the undetected PPP)
Predicted PPP intensity for undetected objects,

$$\lambda_{k+1|k}^{u}(x_{k+1}) = \int p(x_{k+1}|x_{k})P^S(x_{k})\lambda_{k|k}^{u}(x_{k})dx_{k} + \lambda_{k+1}^{B}(x_{k+1})$$

The predicted intensity $\lambda_{k+1|k}^{u}(x_{k+1})$ is the sum of two intensities:

- Prediction of the surviving undetected objects $\int p(x_{k+1}|x_{k})P^S(x_{k})\lambda_{k|k}^{u}(x_{k})dx_{k}$
- Birth of new undetected objects $\lambda_{k+1}^{B}(x_{k+1})$

Mixture representations

$$\lambda_{k+1|k}(x_{k+1}) = \sum_{t=1}^{N_{k}^{u}} w_{k|k}^{u,t} \int p(x_{k+1}|x_{k})P^S(x_{k})\rho_{k|k}^{u,t}(x_{k})dx_{k} + \sum_{i=1}^{N_{k+1}^{B}} w_{k+1}^{B,i}\rho_{k+1}^{B,i}(x_{k+1})$$
### Undetected PPP prediction: pseudo-code

- **Posterior parameters:**
  \[
  \left\{ \left( w_{k|k}^{u,t}, p_{k|k}^{u,t}(\cdot) \right) \right\}_{t=1}^{N_k^u}
  \]

- **Predicted parameters:**
  \[
  \left\{ \left( w_{k+1|k}^{u,t}, p_{k+1|k}^{u,t}(\cdot) \right) \right\}_{t=1}^{N_{k+1}^u}
  = \left\{ \text{Predict} \left( w_{k|k}^{u,t'}, p_{k|k}^{u,t'}(\cdot) \right) \right\}_{t'=1}^{N_k^u} \cup \left\{ \left( w_{k+1}^{B,t''}, p_{k+1}^{B,t''}(\cdot) \right) \right\}_{t''=1}^{N_{k+1}^B}
  \]

Increased number of mixture parameters \( N_{k+1}^u = N_k^u + N_{k+1}^B \)
The predicted PPP weight and density,

$$
(w_{k+1|k}^{u,t}, p_{k+1|k}^{u,t}(\cdot)) = \text{Predict}(w_{k|k}^{u,t}, p_{k|k}^{u,t}(\cdot))
$$

are given by

$$
w_{k+1|k}^{u,t} p_{k+1|k}^{u,t}(x_{k+1}) = w_{k|k}^{u,t} \int \pi_{k+1}(x_{k+1}|x_k) P^S(x_k) p_{k|k}^{u,t}(x_k^i) dx_k
$$

and are

$$
w_{k+1|k}^{u,t} = w_{k|k}^{u,t} P^S_{u,t}
$$

$$
p_{k+1|k}^{u,t}(x_{k+1}) = \int \pi_{k+1}(x_{k+1}|x_k) \frac{P^S(x_k) p_{k|k}^{u,t}(x_k^i)}{P^S_{u,t}} dx_k
$$

where

$$
P^S_{u,t} = \int P^S(x_k) p_{k|k}^{u,t}(x_k) dx_k$$
EXAMPLE: UNDETECTED PPP PREDICTION

Constant $P^S$, linear Gaussian motion model

- $P^S(x) = P^S$ and $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; F_{k+1}x_k, Q_{k+1})$
- PPP birth with Gaussian mixture intensity $\sum_{i=1}^{N^B_{k+1}} w_{k+1,i}^B \mathcal{N}(x_{k+1}; \mu_{k+1}^B, P_{k+1}^B)$
- Posterior intensity $\lambda^u_{k|k}(x_k) = \sum_{t=1}^{N^u_k} w_{k|k}^{u,t} \mathcal{N}(x_k; \mu_{k|k}^u, P_{k|k}^u)$
- Predicted intensity,

$$\lambda^u_{k+1|k}(x_{k+1}) = \sum_{t=1}^{N^u_k} w_{k|k}^{u,t} P^S \mathcal{N}(x_{k+1}; F_{k+1}\mu_{k|k}^u, F_{k+1}P_{k|k}^u F^T_{k+1} + Q_{k+1})$$

$$+ \sum_{i=1}^{N^B_{k+1}} w_{k+1,i}^B \mathcal{N}(x_{k+1}; \mu_{k+1}^B, P_{k+1}^B)$$
$$\lambda^u_{k|k}(x_k) = 0.02\mathcal{N}(x_{k+1}; -16, 3) + 0.03\mathcal{N}(x_{k+1}; -10, 2) + 0.01\mathcal{N}(x_{k+1}; 11, 4) + 0.04\mathcal{N}(x_{k+1}; 19, 1)$$
\[ \lambda_{k|k}(x_k) = \\
0.02 \mathcal{N}(x_{k+1}; -16, 3) + 0.03 \mathcal{N}(x_{k+1}; -10, 2) + 0.01 \mathcal{N}(x_{k+1}; 11, 4) + 0.04 \mathcal{N}(x_{k+1}; 19, 1) \]
EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- \( \lambda^u_{k|k}(x_k) = 0.02 \mathcal{N}(x_{k+1}; -16, 3) + 0.03 \mathcal{N}(x_{k+1}; -10, 2) + 0.01 \mathcal{N}(x_{k+1}; 11, 4) + 0.04 \mathcal{N}(x_{k+1}; 19, 1) \)

- \( P^S = 0.9 \) and random walk, \( \pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; x_k, 4) \)
\[ \lambda_{k|k}(x_k) = 0.02 \mathcal{N}(x_{k+1}; -16, 3) + 0.03 \mathcal{N}(x_{k+1}; -10, 2) + 0.01 \mathcal{N}(x_{k+1}; 11, 4) + 0.04 \mathcal{N}(x_{k+1}; 19, 1) \]

- \( P^S = 0.9 \) and random walk, \( \pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; x_k, 4) \)
\[ \lambda_{\kappa|\kappa}(x_{\kappa}) = 
0.02 \mathcal{N}(x_{\kappa+1}; -16, 3) + 0.03 \mathcal{N}(x_{\kappa+1}; -10, 2) + 0.01 \mathcal{N}(x_{\kappa+1}; 11, 4) + 0.04 \mathcal{N}(x_{\kappa+1}; 19, 1) \]

- \( P^S = 0.9 \) and random walk, \( \pi_{\kappa+1}(x_{\kappa+1}|x_{\kappa}) = \mathcal{N}(x_{\kappa+1}; x_{\kappa}, 4) \)

- \( \lambda_{\kappa+1}(x_{\kappa+1}) = 0.05 \mathcal{N}(x_{\kappa+1}; 0, 1) \)
EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- \( \lambda^u_{k|k}(x_k) = 0.02 \mathcal{N}(x_{k+1}; -16, 3) + 0.03 \mathcal{N}(x_{k+1}; -10, 2) + 0.01 \mathcal{N}(x_{k+1}; 11, 4) + 0.04 \mathcal{N}(x_{k+1}; 19, 1) \)
- \( P^S = 0.9 \) and random walk, \( \pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; x_k, 4) \)
- \( \lambda^B_{k+1}(x_{k+1}) = 0.05 \mathcal{N}(x_{k+1}; 0, 1) \)
EXAMPLE: UNDETECTED PPP PREDICTION VISUALIZATION

- $\lambda^u_{k|k}(x_k) = 0.02\mathcal{N}(x_{k+1}; -16, 3) + 0.03\mathcal{N}(x_{k+1}; -10, 2) + 0.01\mathcal{N}(x_{k+1}; 11, 4) + 0.04\mathcal{N}(x_{k+1}; 19, 1)$
- $P^S = 0.9$ and random walk, $\pi_{k+1}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; x_k, 4)$
- $\lambda^B_{k+1}(x_{k+1}) = 0.05\mathcal{N}(x_{k+1}; 0, 1)$
Detected MBM prediction: pseudo-code

- **Posterior parameters:**
  \[
  \left\{ \left( \ell_{k|k}^h, \left\{ \left( r_{i,k|k}^i, p_{i,k|k}^i(h) \right) \right\}_{i=1}^{N_{h_k}^k} \right) \right\}_{h_k=1}^{\mathcal{H}_k}
  \]

- **Predicted parameters:**
  \[
  \left\{ \left( \ell_{k|k}^h, \left\{ \left( r_{i,k+1|k}^i, p_{i,k+1|k}^i(h) \right) \right\}_{i=1}^{N_{h_k}^k} \right) \right\}_{h_k=1}^{\mathcal{H}_k}
  \]

where, for each \( h_k \) and each \( i \),

\[
\left( r_{i,k+1|k}^i, p_{i,k+1|k}^i(h) \right)
\]

are computed the same way as in an MBM filter

- **Same number of Bernoullis** \( N_{h_k}^k \)
The predicted Bernoulli parameters,
\[
\left( r_{i,h_k}^{i,h_k}, P_{k+1|k}^{i,h_k}(\cdot) \right)
\]
are
\[
r_{k+1|k}^{i,h_k} = r_{k|k}^{i,h_k} P_{i,h_k}^S
\]
\[
P_{k+1|k}^{i,h_k}(x_{k+1}^i) = \int \pi_{k+1}(x_{k+1}^i|x_k^i) P_{k|k}^S(x_k^i) P_{i,h_k}^{i,h_k}(x_k^i) d x_k^i
\]
where
\[
P_{i,h_k}^S = \int P_{k|k}^S(x_k^i) P_{i,h_k}^{i,h_k}(x_k^i) d x_k^i
\]
PMBM PREDICTION: 2D EXAMPLE, $P^S = 0.9$

- Constant velocity $\pi_k(x_k| x_{k-1}) = \mathcal{N}(x_k; Fx_{k-1}, Q)$, and $P^S(x_{k-1}) = 0.9$
- Birth intensity with single mixture component, position in origin, zero velocity
PMBM update: overview

Multi-Object Tracking

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POISSON MULTI-BERNOULLI MIXTURE UPDATE

- **Prior PMBM parameters**
- **Update**

\[ \mathcal{PMBM}_{k|k}(x_k) = \frac{p(z_k|x_k) \mathcal{PMBM}_{k|k-1}(x_k)}{\int p(z_k|x_k') \mathcal{PMBM}_{k|k-1}(x_k') \delta x_k'} \]

with multi-object measurement model \( p(z_k|x_k) \) with

- Probability of detection \( P^D_k(x_k) \)
- Measurement model \( g_k(z_k|x_k) \)
- Poisson clutter intensity: \( \lambda_c(z_k) \)

- **Posterior PMBM parameters**
POISSON MULTI-BERNOULLI MIXTURE UPDATE, IN SUMMARY

PMBM update for the standard point object models

- For each prior MB, multiple data associations
- For each prior MB and each data association, we get an MB in the posterior MBM
- For each Bernoulli, two possibilities:
  - Either associated to one of the measurements,
  - or misdected.
- Any measurement not associated to a prior Bernoulli is either
  - clutter
  - or from an object detected for the first time

We get a new Bernoulli
With an MB birth model,
- Initiation of potential new objects: add birth Bernoulli components in prediction
- When handling the data association, each detection is associated to
  - one of the prior Bernoullis, or
  - the clutter PPP

With a PPP birth model,
- Initiation of potential new objects: measurement driven.
- When handling the data association, each detection is associated to
  - one of the prior Bernoullis,
  - the undetected PPP or the clutter PPP

**Important**: we treat the potential new objects and the clutter jointly!
- Convenient to reformulate assignment problem: assign the measurements
• Let there be $m_k$ measurements and $n_k$ objects.

• The association for measurement $z^j_k$ is denoted $\psi^j_k$.

• An association for all $m_k$ measurements is denoted

$$\psi_k = [\psi^1_k, \psi^2_k, \ldots, \psi^j_k, \ldots, \psi^{m_k}_k]$$

• $\psi^j_k$ is defined similarly to how $\theta^i_k$ was defined,

$$\psi^j_k = \begin{cases} i & \text{if measurement } j \text{ is associated to object } i \\ 0 & \text{if measurement } j \text{ is associated either to a potential new object, or to clutter} \end{cases}$$
DATA ASSOCIATION VARIABLE FOR $M_K$ DETECTIONS, 2

- $\Psi_k$ is the set of valid association events at time $k$.
- For $\psi_k \in \Psi_k$, the following must hold:
  1. Each measurement must be either from a previously detected object, or clutter/potential new object,

     $$\psi^j_k \in \{0, \ldots, n_k\}, \forall j \in \{1, \ldots, m_k\}$$

  2. **Point object assumption**: For any pair of two measurements, they cannot be associated to the same previously detected object,

     $$\forall j, j' \in \{1, \ldots, m_k\}, j \neq j', \text{ if } \psi^j_k \neq 0, \psi^{j'}_k \neq 0 \Rightarrow \psi^j_k \neq \psi^{j'}_k$$
Note that 1. and 2. on the previous slide together implicitly ensures that we do not associate more than $n_k$ measurements to the $n_k$ objects.

In what follows, unless otherwise stated, we consider associations $\psi_k \in \Psi_k$.

Given a $\psi$, we can find the equivalent $\theta$, and vice versa.

\[
\psi^j = i \iff \theta^i = j \\
\psi^j = 0 \iff \nexists \ i : \theta^i = j \\
\nexists \ j : \psi^j = i \iff \theta^i = 0
\]
HYPOTHESIS ORIENTED-PMBM UPDATE

Hypothesis oriented-PMBM update: pseudo-code

**Input:** \( \lambda_{k|k-1}^u(x_k), \{ \left( \ell_{h_{k-1}}, \left\{ \left( r_{i,h_{k-1}}^i, h_{k-1} \right), p_{i,h_{k-1}}^i(\cdot) \right\}_{i=1}^{N_{k-1}^h} \right) \}_{h_{k-1}=1}^{H_{k-1}} \)

Mis detection update: \( \lambda_{k|k}^u(x_k) = (1 - P^D(x_k)) \lambda_{k|k-1}^u(x_k) \)

Initialise: \( h_k = 0 \)

For \( h_{k-1} = 1, \ldots, H_{k-1} \)

Create cost matrix \( L_{h_{k-1}}^{h_{k-1}} \), and compute \( M_{h_{k-1}} \) associations \( \psi_m \)

For \( m = 1, \ldots, M_{h_{k-1}} \)

Increase: \( h_k \leftarrow h_k + 1 \)

**Compute posterior MB parameters:** detected, misdetected & new Bernoulli, log-weight \( \tilde{\ell}_{h_k} \)

Set \( H_k = h_k \)

Normalise log-weights \( \ell_{h_k} \leftarrow \tilde{\ell}_{h_k} \)

**Output:** \( \lambda_{k|k}^u(x_k), \{ \left( \ell_{h_k}, \left\{ \left( r_{i,h_k}^i, h_k \right), p_{i,h_k}^i(\cdot) \right\}_{i=1}^{N_k^h} \right) \}_{h_k=1}^{H_k} \)
PMBM update: details

Multi-Object Tracking

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Undetected object: remain undetected, or detected for the first time

Previously detected object: misdetected, or detected again

Important “building blocks” of PMBM update:

- update of PPP intensity for undetected objects that remain undetected
- update of potential new object detected for the first time ⇒ new Bernoulli
- update of Bernoulli with associated measurement – similar to MBM-filter
- update of misdetected Bernoulli – similar to MBM-filter
- posterior log-weights
- Posterior PPP intensity for objects that remain undetected,
  \[ \lambda^u_{k|k}(x) = (1 - P^D(x)) \lambda^u_{k|k-1}(x) \]

- Posterior intensity is **lower/higher** in areas where \((1 - P^D(x))\) is **low/high**, because it is **unlikely/likely** that an object was not detected there.

- Mixture representation of intensity
  \[
  \lambda^u_{k|k}(x_k) = \sum_t w^u_{k|k-1} (1 - P^D(x)) p^u_{k|k-1}(x_k)
  \]
  \[
  = \sum_t w^u_{k|k-1} P^MD_{u,t} \frac{(1 - P^D(x)) p^u_{k|k-1}(x_k)}{P^MD_{u,t}} = \sum_t \left( w^u_{k|k-1} \right) \frac{(1 - P^D(x)) p^u_{k|k-1}(x_k)}{P^MD_{u,t}}
  \]

where \(P^MD_{u,t} = \int (1 - P^D(x)) p^u_{k|k-1}(x_k) dx_k\)
Constant $P^D$, linear Gaussian models

- Prior intensity $\lambda_{k|k-1}^u(x_k) = \sum_{t=1}^{5} w_{k|k-1}^{u,t} \mathcal{N}(x_k; \mu_{k|k-1}^{u,t}, P_{k|k-1}^{u,t})$
- Probability of detection $P^D = 0.75$
- Posterior intensity $\lambda_{k|k}^u(x_k) = \sum_{t=1}^{5} 0.25 w_{k|k-1}^{u,t} \mathcal{N}(x_k; \mu_{k|k-1}^{u,t}, P_{k|k-1}^{u,t})$
If $\psi^j = 0$, then the measurement $z^j_k$ is either clutter or from a previously undetected object.

New Bernoulli component in the posterior MB, with parameters

$$r^j_{k|k} = \frac{\rho^u_{k|k-1}(z^j_k)}{\lambda_c(z^j_k) + \rho^u_{k|k-1}(z^j_k)},$$

$$\rho^u_{k|k-1}(z^j_k) = \int P^D(x_k)g_k(z^j_k|x_k)\lambda^u_{k|k-1}(x_k)dx_k$$

$$p^j_{k|k}(x_k) = \frac{P^D(x_k)g_k(z^j_k|x_k)\lambda^u_{k|k-1}(x_k)}{\rho^u_{k|k-1}(z^j_k)}$$

**Posterior $r$ conditioned on $\psi$**

Relative intensity of: 1) detection from previously undetected object; and, 2) clutter

Predicted log-likelihood

$$\ell^u_{k} = \log \left( \lambda_c(z^j_k) + \rho^u_{k|k-1}(z^j_k) \right)$$
Earlier we had that, conditioned on the data association, $r = 1$

How can we now have a new Bernoulli for which $r < 1$?

It represents two possibilities:

- Detection was from clutter – no new object
  Likelihood $\lambda_c(z_k^i)$.
- Detection was from a new object
  Likelihood $\rho^u_{k|k-1}(z_k^i) = \int P^D(x_k)g_k(z_k^i|x_k)\lambda^u_{k|k-1}(x_k)dx_k$.

Represented compactly as a new Bernoulli with probability of existence

$$ r_{k|k}^j = \frac{\rho^u_{k|k-1}(z_k^i)}{\lambda_c(z_k^i) + \rho^u_{k|k-1}(z_k^i)} $$
Constant $P^D$, linear Gaussian models

- Undetected intensity $\lambda^u_{k|k-1}(x_k) = \sum_{t=1}^5 w^u_{k|k-1} \mathcal{N}(x_k; \mu^u_{k|k-1}, P^u_{k|k-1})$

- Probability of detection $P^D = 0.9$, measurement model $g_k(z|x) = \mathcal{N}(z; x, R)$

- Likelihood $\rho^u_{k|k-1}(z_k) = \sum_{t=1}^5 P^D w^u_{k|k-1} \mathcal{N}(z_k; \mu^u_{k|k-1}, P^u_{k|k-1} + R)$

- Clutter $\lambda_c(z_k) = \bar{\lambda}_c / V$

- Probability of existence of new Bernoulli

$$r_{k|k} = \frac{\sum_{t=1}^5 P^D w^u_{k|k-1} \mathcal{N}(z_k; \mu^u_{k|k-1}, P^u_{k|k-1} + R)}{\bar{\lambda}_c/V + \sum_{t=1}^5 P^D w^u_{k|k-1} \mathcal{N}(z_k; \mu^u_{k|k-1}, P^u_{k|k-1} + R)}$$
NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE
NEW BERNOULLI PROBABILITY OF EXISTENCE EXAMPLE

\[ \rho_{k|k-1}^u(z_k) \]

\[ \lambda_c(z_k) \]

\[ r_{k|k} = \frac{\rho_{k|k-1}^u(z_k)}{\lambda_c(z_k) + \rho_{k|k-1}^u(z_k)} \]
New Bernoulli State Density Example

Constant $P^D$, linear Gaussian models

- Undetected intensity $\lambda^u_{k|k-1}(x_k) = \sum_t w^u_{k|k-1} \mathcal{N}(x_k; \mu^u_{k|k-1}, P^u_{k|k-1})$

- Probability of detection $P^D = 0.9$, measurement model $g_k(z|x) = \mathcal{N}(z; Hx, R)$

- State density of new Bernoulli

$$p^j_k(x_k) = \sum_t w^u_{k|k-1} \mathcal{N}(x_k; \mu^u_{k|k-1} + K^u_k (z^j_k - \hat{z}^u_k), P^u_{k|k-1} - K^u_k H_k P^u_{k|k-1})$$

$$w^u_{k,t,j} = \frac{w^u_{k|k-1} P^D \mathcal{N}(z^j_k; \hat{z}^u_k, S_k)}{\sum_{t'} w^u_{k|k-1} P^D \mathcal{N}(z^j_k; \hat{z}^{u,t'}_k, S^{u,t'}_k)}$$

- Pruning and merging used to reduce $p^j_k(x_k)$, often to a single Gaussian
NEW BERNOULLI STATE DENSITY VISUALIZATION, 1

Posterior GM
Posterior GM, after pruning, threshold $10^{-4}$
NEW BERNOULLI STATE DENSITY VISUALIZATION, 1

Posterior GM, after pruning and merging
Prior and posterior
Prior intensity components
NEW BERNULLI STATE DENSITY VISUALIZATION, 2

Prior and posterior
BERNOULLI UPDATE: DETECTION AND MISDETECTION

For prior hypothesis $h$, if $\psi^j = i$,

\[
r_{i,j,h}^{k|k} = 1
\]

\[
p_{i,j,h}^{k|k}(x_k^i) = \frac{P^D(x_k^i)g_k(z_k^i|x_k^i)p_{i,h}^{k|k-1}(x_k^i)}{\int P^D(x_k^i)g_k(z_k^i|x_k^i)p_{i,h}^{k|k-1}(x_k^i)dx_k^i}
\]

Predicted log-likelihoods

\[
\ell_{k}^{i,j,h} = \log \left( r_{k|k-1}^{i,h} \int P^D(x_k^i) g_k(z_k^i|x_k^i) p_{k|k-1}^{i,h}(x_k^i) dx_k^i \right)
\]

\[
\ell_{k}^{i,0,h} = \log \left( 1 - r_{k|k-1}^{i} + r_{k|k-1}^{i} P_{i,h}^{MD} \right)
\]

Note similarity to Bernoulli update in MBM filter

For prior hypothesis $h$, if $\nexists j : \psi^j = i$,

\[
r_{i,0,h}^{k|k} = \frac{r_{k|k-1}^{i,h} P_{i,h}^{MD}}{1 - r_{k|k-1}^{i,h} + r_{k|k-1}^{i,h} P_{i,h}^{MD}}
\]

\[
p_{i,0,h}^{k|k}(x) = \frac{(1 - P^D(x_k)) p_{k|k-1}^{i,h}(x)}{P_{i,h}^{MD}}
\]

where $P_{i,h}^{MD} = \int (1 - P^D(x_k)) p_{k|k-1}^{i,h}(x_k) dx_k^i$
For a prior MB $h$ and a data association $\psi_k$, the non-normalized posterior log-weight is

$$
\hat{\ell}_{k|k}^{h_{k-1}, \psi_k} = \ell_{k|k-1}^{h_{k-1}} + \sum_{i: \# j: \psi^j = i} \ell_{k}^{i, 0, h} + \sum_{j: \psi^j \neq 0} \ell_{k}^{\psi^j, j, h} + \sum_{j: \psi^j = 0} \ell_{k}^{u, j}
$$

where

$$
\hat{\ell}_{k}^{j, h} = \begin{cases} 
\ell_{k}^{\psi^j, j, h} - \ell_{k}^{\psi^j, 0, h} & \text{if } \psi^j \neq 0 \\
\ell_{k}^{u, j} & \text{if } \psi^j = 0 
\end{cases}
$$

Prior

Misdetection

Assoc. meas.

Clutter or potential new object

$$
= \ell_{k|k-1}^{h_{k-1}} + \sum_{i=1}^{N^h} \ell_{k}^{i, 0, h} + \sum_{j: \psi^j \neq 0} \left[ \ell_{k}^{\psi^j, j, h} - \ell_{k}^{\psi^j, 0, h} \right] + \sum_{j: \psi^j = 0} \ell_{k}^{u, j}
$$

$$
= \ell_{k|k-1}^{h_{k-1}} + \sum_{j=1}^{m_k} \hat{\ell}_{k}^{j, h} + \text{Constant independent of } \psi_k
$$
Prior and model, 2D scenario

- PMBM with two MBs, each with two Bernoullis with Gaussian state densities, undetected PPP intensity with single Gaussian.
- Measurement model:
  \[
  P^D = 0.75 \\
  g_k(z|x) = \mathcal{N}(z; x, 9I_2) \\
  \lambda_c(z) = \begin{cases} 
  4 \times 10^{-4} & z \in [-25, 25] \times [-25, 25] \\
  0 & z \notin [-25, 25] \times [-25, 25]
  \end{cases}
  \]
- Single measurement \(\Rightarrow\) 3 DAs for each prior MB \(\Rightarrow\) Posterior MBM with 6 MBs
Undetected objects PPP (Orange), Detected objects MB (Blue), Measurement (Red)

Prior MB weight = 0.3

Prior MB weight = 0.7

Post. MB weight = 0.01

Post. MB weight = 0.05

Post. MB weight = 0.17

Post. MB weight = 0.01

Post. MB weight = 0.64

Post. MB weight = 0.11
The data association problem is handled analogously to tracking \( n \) objects, and MBM filter:

- Use gating to remove very unlikely associations and group Bernoullis/measurements
- For each group, form cost matrix with negative log likelihoods
- Use some algorithm to find \( M \) associations, e.g.,
  - Murty
  - Gibbs’ sampling
- Truncate all other associations
Let there be $m_k$ detections, and consider an MB $h$ with $N^h$ Bernoullis. The cost matrix is

\[
L^h = \begin{bmatrix}
-\ell_{1,1}^h & -\ell_{1,2}^h & \cdots & -\ell_{1,N^h}^h & -\ell_{1,0}^h & \infty & \cdots & \infty \\
-\ell_{2,1}^h & -\ell_{2,2}^h & \cdots & -\ell_{2,N^h}^h & \infty & -\ell_{2,0}^h & \cdots & \infty \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\ell_{m_k,1}^h & -\ell_{m_k,2}^h & \cdots & -\ell_{m_k,N^h}^h & \infty & \infty & \cdots & -\ell_{m_k,0}^h
\end{bmatrix}
\]

where

\[
\ell_{j,0}^i = \log \left( \lambda_c(z_k^i) + \int P^D(x_k) g_k(z_k^i|x_k) \lambda^u_{k|k-1}(x_k) \, dx_k \right)
\]

\[
\ell_{j,i}^h = \log \left( \frac{r_{k|k-1}^i} {1 - r_{k|k-1}^i + r_{k|k-1}^i \int (1 - P^D(x_k^i)) \, dx_k^i} \right) - \log \left( \frac{\int P^D(x_k^i) g_k(z_k^i|x_k^i) p_{k|k-1}^i(x_k^i) \, dx_k^i} {r_{k|k-1}^i} \right)
\]
PMBM post processing

Multi-Object Tracking

Karl Granström
After prediction and update, we have a PMBM density $\mathcal{PMBM}_{k|k}(x_k)$ with params.

\[
\left\{ \left( w_{k|k}^u, p_{k|k}^u(\cdot) \right) \right\}_{t=1}^{N_k^u}, \left\{ \left( \ell_{k|k}^h, \left\{ \left( r_{i,k}^h, p_{i,k}^h(\cdot) \right) \right\}_{i=1}^{N_k^h} \right) \right\}_{h_k=1}^{\mathcal{H}_k}
\]

- **Reduction:**
  - Reduce $N_k^u$, $\mathcal{H}_k$ and $N_k^h$
  - Important for computational cost
  - Pruning, merging, capping, and **recycling**

- **Estimation:**
  - Extracting a set of estimated object states from the posterior density.
PMBM reduction

- **MBM pruning**: prune MB $h_k$ if $\ell_{k|k}^{h_k} \leq \Gamma$

- **MBM capping**: if $H_k > N_{\text{max}}$, keep the $N_{\text{max}}$ MBs with largest log-weights.

  After pruning and capping the MBM, remaining log-weights are re-normalized.

- **Bernoulli recycling**: in each MB $h_k$, recycle Bernoulli $i$ if $r_{k|k}^{i;h} < \Gamma^r$

- **PPP reduction**: pruning, merging and capping of the mixture intensity

Outside the scope of the course: MBM merging
Bernoulli recycling: basic idea

Instead of pruning Bernoullis with small $r$, approximate them as PPP, and add the intensity to undetected PPP intensity.

- KL-div minimising PPP approximation: intensity $= \text{Bernoulli PHD},$

$$\lambda_{k|k}^{h,i,\text{REC}}(x_k) = r_{k|k}^{h,i} p_{k|k}^{h,i}(x_k)$$

- Undetected PPP intensity after recycling is

$$\lambda_{k|k}^{u,\text{REC}}(x_k) = \lambda_{k|k}^{u}(x_k) + \sum_h \sum_{i:r_{k|k}^{h,i} < \Gamma} \exp\left(\ell_{k|k}^{h,i}\right) r_{k|k}^{h,i} p_{k|k}^{h,i}(x_k)$$

Note that we must take the normalized hypothesis weight $\exp\left(\ell_{k|k}^{h,i}\right)$ into account
WHY RECYCLING?

Following the recycling, we have to reduce the undetected PPP intensity.

**Why not just prune right away?**

- Recycling/pruning lowers the computational complexity, because there are fewer Bernoullis to consider in the data association.
- Pruning means that we lose all information contained in what is pruned.
- By recycling, the information is retained approximately as a PPP.
- The Bernoulli recycling threshold can therefore be considerably larger than a Bernoulli pruning threshold.
- Empirical studies show that Bernoulli recycling leads to lower computational cost, without sacrificing tracking performance.
REDUCING THE UNDETECTED INTENSITY

- Undetected intensity with mixture representation

\[ \lambda_{k|k}^{u}(x_k) = \sum_{t=1}^{N_k^u} w_{k|k}^{u,t} p_{k|k}^{u,t}(x_k) \]

- The number of mixture components increases over time, due to:
  - the addition of birth in the prediction
  - the recycling following the update

- Reduced using pruning, merging and capping.
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\text{max}} = 2$
- Re-normalize weights
- Bernoulli recycling, $\Gamma_r = 0.1$
- PPP pruning $\Gamma_w = 0.05$

Note: typically, $\Gamma$ and $\Gamma_w$ are smaller, and $N_{\text{max}}$ is larger.
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$

![Graphs showing posterior intensity and MB weights](image-url)
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$
- MBM capping, $N_{\text{max}} = 2$

\[
\Gamma_r = 0.1
\]
\[
\Gamma_w = 0.05
\]

Post. MB weight = 0.067

Post. MB weight = 0.08

Post. MB weight = 0.736

Post. MB weight = 0.101
EXAMPLE PMBM REDUCTION

- MBM pruning, $\Gamma = \log(0.05)$

- MBM capping, $N_{\text{max}} = 2$

![Posterior intensity](image1.png)

- Post. MB weight = 0.736

![Posterior intensity](image2.png)

- Post. MB weight = 0.101
EXAMPLE PMBM REDUCTION

- MBM pruning, \( \Gamma = \log(0.05) \)
- MBM capping, \( N_{\text{max}} = 2 \)
- Re-normalize weights

Posterior intensity

Post. MB weight = 0.736

Post. MB weight = 0.101
• MBM pruning, 
  \[ \Gamma = \log(0.05) \]
• MBM capping, 
  \[ N_{\text{max}} = 2 \]
• Re-normalize weights
EXAMPLE PMBM REDUCTION

- MBM pruning,  \( \Gamma = \log(0.05) \)
- MBM capping,  \( N_{\text{max}} = 2 \)
- Re-normalize weights
- Bernoulli recycling,  \( \Gamma' = 0.1 \)
EXAMPLE PMBM REDUCTION

- MBM pruning,
  $\Gamma = \log(0.05)$
- MBM capping,
  $N_{\text{max}} = 2$
- Re-normalize weights
- Bernoulli recycling,
  $\Gamma' = 0.1$

![Graph showing after recycling and post. MB weight](image)

After recycling

Post. MB weight = 0.879

Post. MB weight = 0.121

Note: typically, $\Gamma$ and $\Gamma w$ are smaller, and $N_{\text{max}}$ is larger.
EXAMPLE PMBM REDUCTION

- MBM pruning, \( \Gamma = \log(0.05) \)
- MBM capping, \( N_{\text{max}} = 2 \)
- Re-normalize weights
- Bernoulli recycling, \( \Gamma^r = 0.1 \)
- PPP pruning \( \Gamma^w = 0.05 \)

\[ \text{Post. MB weight} = 0.879 \]
\[ \text{Post. MB weight} = 0.121 \]
EXAMPLE PMBM REDUCTION

- MBM pruning, \( \Gamma = \log(0.05) \)
- MBM capping, \( N_{\text{max}} = 2 \)
- Re-normalize weights
- Bernoulli recycling, \( \Gamma^r = 0.1 \)
- PPP pruning \( \Gamma^w = 0.05 \)
EXAMPLE PMBM REDUCTION

- MBM pruning, \( \Gamma = \log(0.05) \)
- MBM capping, \( N_{\text{max}} = 2 \)
- Re-normalize weights
- Bernoulli recycling, \( \Gamma^r = 0.1 \)
- PPP pruning \( \Gamma^w = 0.05 \)

**Note:** typically, \( \Gamma \) and \( \Gamma^w \) are smaller, and \( N_{\text{max}} \) is larger
Simple PMBM estimator

Generally we do not extract estimates from the undetected PPP.

- Initialise as empty set: \( \hat{x}_{k|k} = \emptyset \)
- MB with largest weight: \( h^* = \max_h \ell_{k|k}^h \)
- For \( i = 1, \ldots, N_{k|k}^{h^*} \), if \( r_{k|k}^{h^*,i} > \Gamma^e \): \( \hat{x}_{k|k} \leftarrow \hat{x}_{k|k} \cap \hat{x}_{k|k}^{i,h^*} \)
- For example, expected value or MAP estimate,

\[
\bar{x}_{k|k}^{i} = \int x_k p_{k|k}^{h^*,i}(x_k)dx_k, \quad \hat{x}_{k|k}^{i,\text{MAP}} = \arg \max_{x_k} p_{k|k}^{h^*,i}(x_k)
\]
EXAMPLE PMBM ESTIMATOR

- Largest weight
- Extract: $\Gamma^e = 0.5$
Implementation of Conjugate multi-object filters

Multi-Object Tracking

Karl Granström
HO-MHT AND TO-MHT

- Two MHT approaches to tracking $n$ objects:
  - Hypothesis oriented (HO): represent each global $n$ object hypothesis explicitly
  - Track oriented (TO): represent each object by local hypotheses. Global $n$ object hypotheses encoded by look-up table.
  - Track oriented is computationally more efficient.

- Similar alternatives for the MB mixture in PMBM and MBM filtering
  - Hypothesis oriented: represent each MB explicitly
  - Track oriented: represent each Bernoulli by local hypotheses. Each MB (global hypothesis) encoded by look-up table.
  - Again, track oriented is computationally more efficient.
<table>
<thead>
<tr>
<th>MBM filter</th>
<th>PMBM filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Add new Bernoullis in prediction</td>
<td>• Initiate new Bernoullis in update</td>
</tr>
<tr>
<td>• New local hypotheses in update</td>
<td>• New local hypotheses in update</td>
</tr>
<tr>
<td>• Look-up table for MBs</td>
<td>• Look-up table for MBs</td>
</tr>
<tr>
<td>• Reduction: remove local hypotheses and global hypotheses</td>
<td>• Reduction: remove local hypotheses and global hypotheses</td>
</tr>
</tbody>
</table>
• Local and global hypotheses

• Bernoulli representation in the MBs
  - Uncertain existence, $r \in (0, 1)$
  - Certain existence, $r = 0$ or $r = 1$
Local and Global Hypotheses in MBM filter

Multi-Object Tracking

Karl Granström
Example: scenario setup

- MB birth with a single birth component in each time step
- Three time steps, measurement sets:
  - $z_1 = \{ z_1^1, z_1^2 \}$
  - $z_2 = \emptyset$
  - $z_3 = \{ z_3^1 \}$
- At time $k = 0$, empty MBM. Not necessary, but most common.
For each leaf node, we have $r$ and $p(x)$ conditioned on that association sequence.
Total number of MBs (global hypotheses): 12

For simplicity: local hypotheses indexed 1, 2, 3 ... from left to right.
Add new Bernoullis in the prediction

For each Bernoulli, maintain local hypotheses

Look-up table points out which local hypotheses are included in an MB

MBM reduction affects both the local hypotheses and look-up table
Local and Global Hypotheses in PMBM filter

Multi-Object Tracking

Karl Granström
Example: scenario setup

- PPP birth, i.e., initiation of new Bernoullis is measurement driven
- Three time steps, measurement sets:
  - $z_1 = \{ z_1^1, z_1^2 \}$
  - $z_2 = \emptyset$
  - $z_3 = \{ z_3^1 \}$
- At time $k = 0$, empty MBM. Not necessary, but most common.
For each leaf node, we have $r$ and $p(x)$ conditioned on that association sequence.
PMBM GLOBAL HYPOTHESES

Total number of MBs (global hypotheses): 3

Look-up table:

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 2 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

For simplicity: local hypotheses indexed from left to right.

Note: each Bernoulli is not represented in each MB (global hypothesis)
- Add a new Bernoulli for each measurement
- For each Bernoulli, maintain local hypotheses
- Look-up table points out which local hypotheses are included in an MB
- MBM reduction affects both the local hypotheses and look-up table
Reduction of local and global hypotheses

Multi-Object Tracking

Karl Granström
Global hypotheses are removed

Before pruning/capping

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 2 & 1 \\
2 & 1 & 1 \\
3 & 1 & 1 \\
3 & 1 & 2 \\
3 & 2 & 1 \\
4 & 1 & 1 \\
5 & 1 & 1 \\
5 & 1 & 2 \\
5 & 2 & 1 \\
6 & 1 & 1 \\
\end{pmatrix}
\]
Global hypotheses are removed

Four global hypotheses are pruned/capped
Some local hypotheses no longer included in an MB
Global hypotheses are removed

Prune un-used local hypotheses, adjust look-up table
If a Bernoulli has no local hypothesis included in any global hypothesis, naturally it can be pruned entirely.
Local hypotheses are removed

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 2 & 1 \\
2 & 1 & 1 \\
3 & 1 & 1 \\
3 & 1 & 2 \\
3 & 2 & 1 \\
4 & 1 & 1 \\
\end{bmatrix}
\]
Local hypotheses are removed

Example: prune local hypothesis corresponding to 2 misdetections
Local hypotheses are removed

Adjust look-up table
Global hypotheses should be unique

Before reduction:

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
2 & 1 & 1 \\
3 & 1 & 1 \\
3 & 2 & 1 \\
4 & 1 & 1
\end{bmatrix}
\]

After reduction:

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 1 \\
2 & 0 & 1 \\
3 & 0 & 1 \\
3 & 0 & 1 \\
4 & 0 & 1
\end{bmatrix}
\]

Unique global hypotheses:

\[
\begin{bmatrix}
1 & 1 \\
2 & 1 \\
3 & 1 \\
4 & 1
\end{bmatrix}
\]

Important to adjust the weights accordingly.
MBs with certain object existence

Multi-Object Tracking

Karl Granström
In both MBM and PMBM there are Bernoullis with $0 < r < 1$.

Expected cardinality is $r$

- $r = 0.5 \Rightarrow$ we expect half an object
- For example, how should an autonomous car react if there is half a car in front?

In reality, an object is either there, or not.

Compare to $n$ object tracking: an integer number of objects, no fractions of objects

Can we have a multi-object density such that each hypothesis represents an integer number of objects?

Yes, if we change to a so called MBM$_{01}$ representation.

The $\delta$-GLMB filter can be interpreted as having a MBM$_{01}$ representation.
A Bernoulli \((r, p(\cdot))\) with \(r \in (0, 1)\) represents two possibilities,

- With probability \(r\) we have exactly one object, with state pdf \(p(\cdot)\).
- With probability \(1 - r\), we have exactly zero objects.

We can represent this as a Bernoulli mixture density,

\[
p(x) = rB^1(x) + (1 - r)B^2(x)
\]

where \(B^1(x)\) and \(B^2(x)\) are Bernoulli densities with parameters

\[
(r^1, p^1(\cdot)) = (1, p(\cdot)), \quad (r^2, p^2(\cdot)) = (0, \text{any pdf})
\]

Instead of one Bernoulli with uncertain existence \(r\), we have two hypotheses that each have certain existence: either one object \((r^1 = 1)\) or zero objects \((r^2 = 0)\).
An MB with two Bernoullis \((r^1, p_1(\cdot))\) and \((r^2, p_2(\cdot))\), with \(r^1 \in (0, 1)\) and \(r^2 \in (0, 1)\) corresponds to \(2^2 = 4\) hypotheses with certain object existence:

### Example: MB with two Bernoullis

Each hypothesis is a special kind of MB:

1) Zero Bernoullis, 2) & 3) One Bernoulli, \(r = 1\), 4) Two Bernoullis, \(r = 1\)
EXPANDING AN MB TO CERTAIN EXISTENCE

- Consider an MB with \( n \) Bernoullis, with parameters \( (r^i, p^i(\cdot)) \).
- Let \( n' \leq n \) of the Bernoullis have \( r \in (0, 1) \), and let remaining \( n - n' \) have \( r = 1 \).
- Can be expanded into a MBM_{01} with \( 2^{n'} \) MBs, where each Bernoulli has \( r = 1 \).

Example: MB with \( n = 3 \), \( n' = 2 \) leads to MBM_{01} with \( 2^{n'} = 4 \) MBs
MBM WITH CERTAIN EXISTENCE

- We denote this type of MBM as MBM\(_{01}\).
- Any MB(M) can be expanded into an MBM\(_{01}\).
- Not all MBM\(_{01}\) have a simpler MB equivalent.

**MBM\(_{01}\) with two equally probable components**

Cannot be simplified to MB with two Bernoullis
WHEN IS THIS USEFUL?

- MBM$_{01}$ representation can feel more intuitive, with an integer number of objects in each global hypothesis.

- Unusually specific birth model: If we know that new objects appear in the surveillance area together in groups, e.g., in pairs, the birth can be modeled as MBM$_{01}$.

- Facilitates other multi-object estimator:
  - Find MAP cardinality estimate
  - Find most probable global hypothesis with this cardinality
  - Extract estimates

- However, worse computational cost with MBM$_{01}$ representation.
Objects may appear and disappear:
MB$_{01}$ with $N$ Bernoullis, birth with $N^B$ Bernoullis $\Rightarrow$ predicted MBM$_{01}$ with $2^{N+N^B}$ hyps.

One object (blue), $P^S = 0.95$, and one birth (orange), $r^B = 0.01$

- Intractable in practice for large $N$ and $N^B$, approximations are required.
- Regular MB prediction does not require approximation.
Posterior MBM$_{01}$ has many more components, compared to posterior MBM

Update with $m_k$ measurements

Number of data associations:

MB : $N_A(m_k, 2)$
MBM$_{01}$ : $N_A(m_k, 0) + 2N_A(m_k, 1) + N_A(m_k, 2)$
SUMMARY OF MB VS MBM\textsubscript{01}

With an MBM\textsubscript{01} representation:

- Requires additional approximations in both the prediction and the update
- Generally requires a higher number of global hypotheses to achieve the same performance

Simulation studies have shown that the MBM\textsubscript{01} representation results in a higher computational cost to achieve the same tracking performance.